Geometric structure and transversal logic of quantum Reed–Muller codes

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Summary

- Quantum Reed–Muller (QRM) codes are a family of CSS codes constructed from the classical Reed–Muller family.
- We give a geometric definition of QRM codes using the structure of an m-dimensional hypercube.
- By studying natural "subcube" transversal operators on these codes, we give:

(1) Necessary and sufficient conditions for when these operators preserve the codespace.

(2) A combinatorial desciption of the logical circuits they implement.

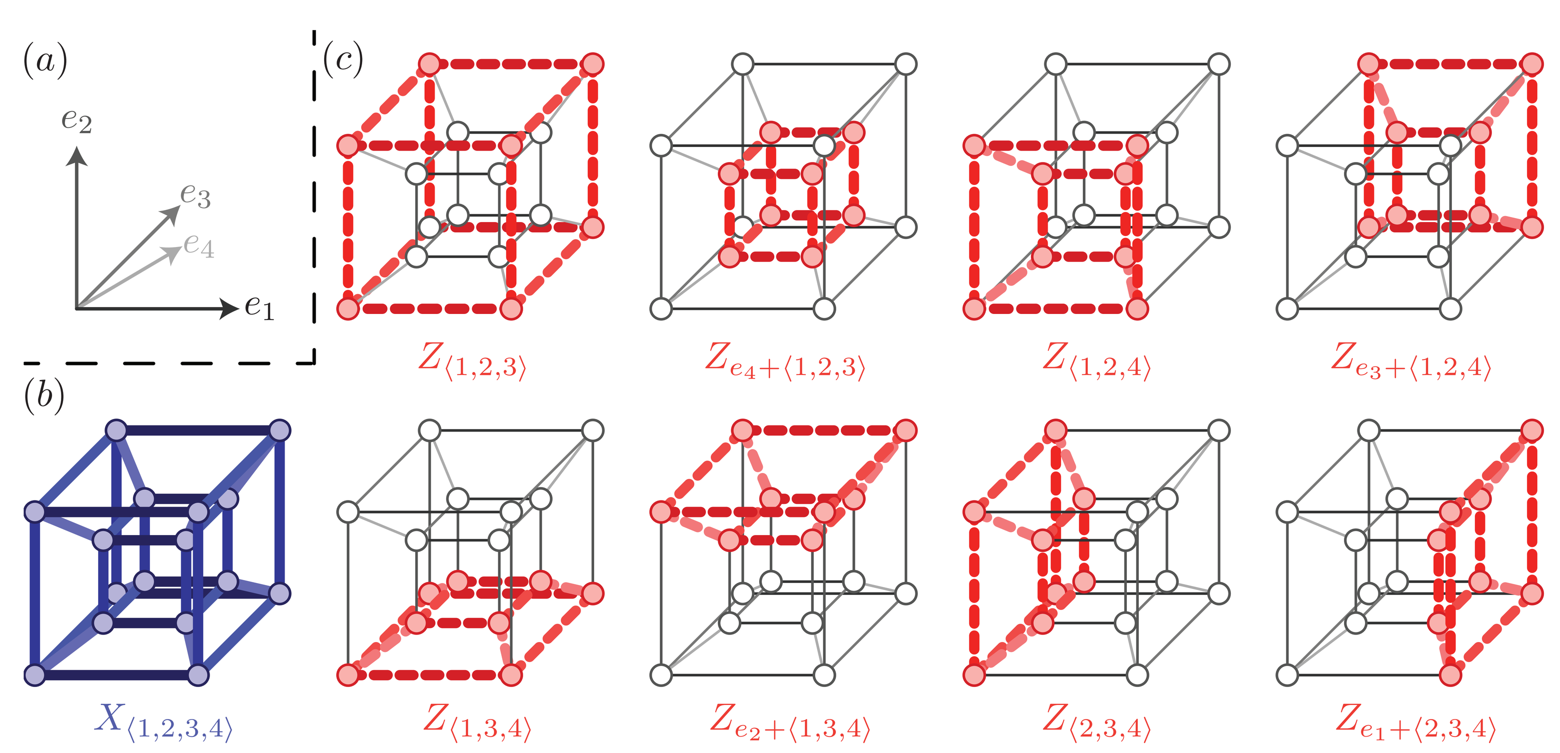
The Geometry of QRM Codes

Let $S = \{e_i\}_{i=1}^{m}$ be the set of weight-1 bit strings of length m.

The m-dimensional hypercube is the Cayley graph $Cay(\mathbb{Z}_2^m, S)$ and a subcube of dimension $|K|$ is any coset of the form $x + \langle K\rangle$ for $K \subseteq S$.

For integers $-1 \le q \le r \le m$, the QRM code $QRM_m(q,r)$ has stabilizers generated by:

$$S_Z \equiv \{Z_{x+\langle K\rangle} \mid K \subseteq S,\ |K| = r+1\},$$
$$S_X \equiv \{X_{x+\langle K\rangle} \mid K \subseteq S,\ |K| = m-q\}.$$

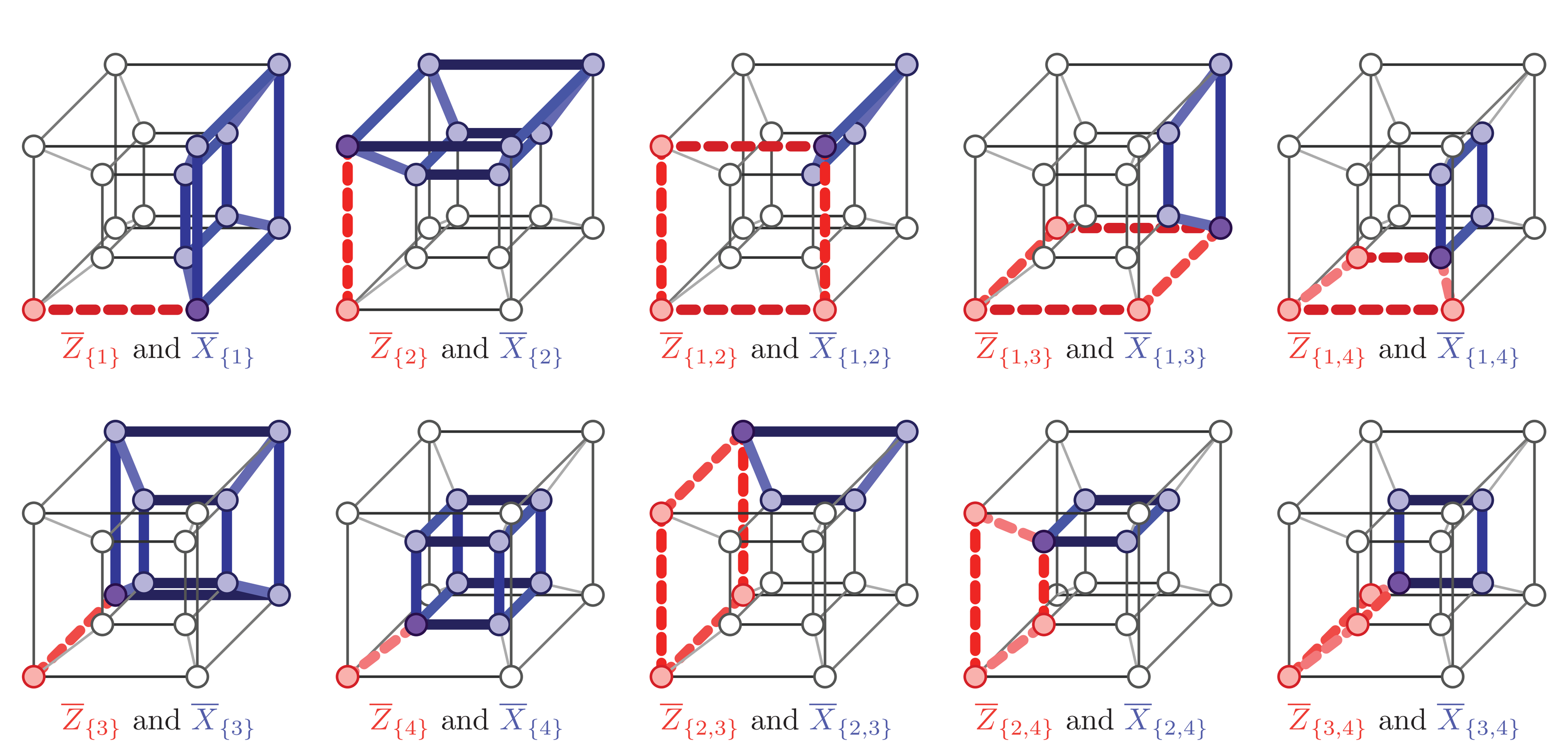


Physical and Logical Operators

Like the stabilizers, a basis for the logical Pauli operators of $QRM_m(q,r)$ is given by Z/X operators acting on subcubes.

Defining the set $\mathcal{Q} \equiv \{J \subseteq S \mid q+1 \le |J| \le r\}$, a basis is given by the operators:

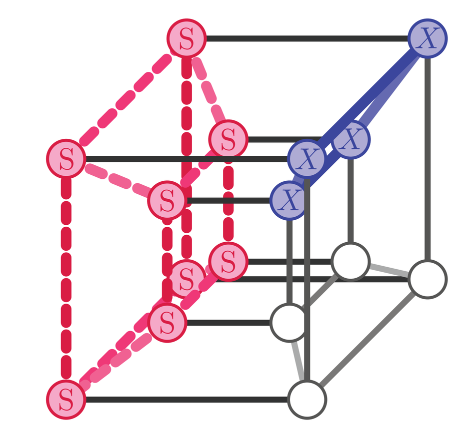
$$L_Z \equiv \{\overline{Z}_J = Z_{\langle J\rangle} \mid J \in \mathcal{Q}\},$$
$$L_X \equiv \{\overline{X}_J = X_{1^m+\langle J\rangle} \mid J \in \mathcal{Q}\}.$$



Subcube Operators

Define $Z(k) \equiv |0\rangle\langle 0| + e^{i\frac{\pi}{2^k}}|1\rangle\langle 1|$

We consider $Z(k)$ operators acting on subcubes, e.g., $S_{\langle 2,3,4\rangle}$, $X_{1111+\langle 3,4\rangle}$.

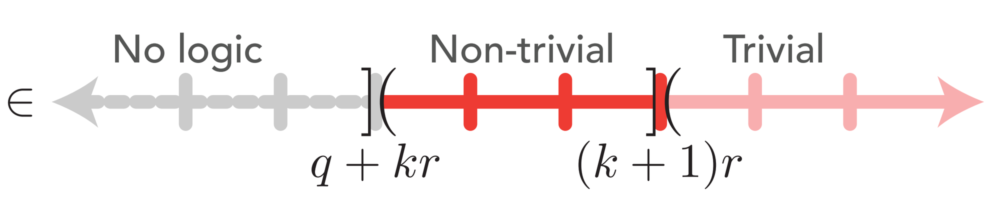


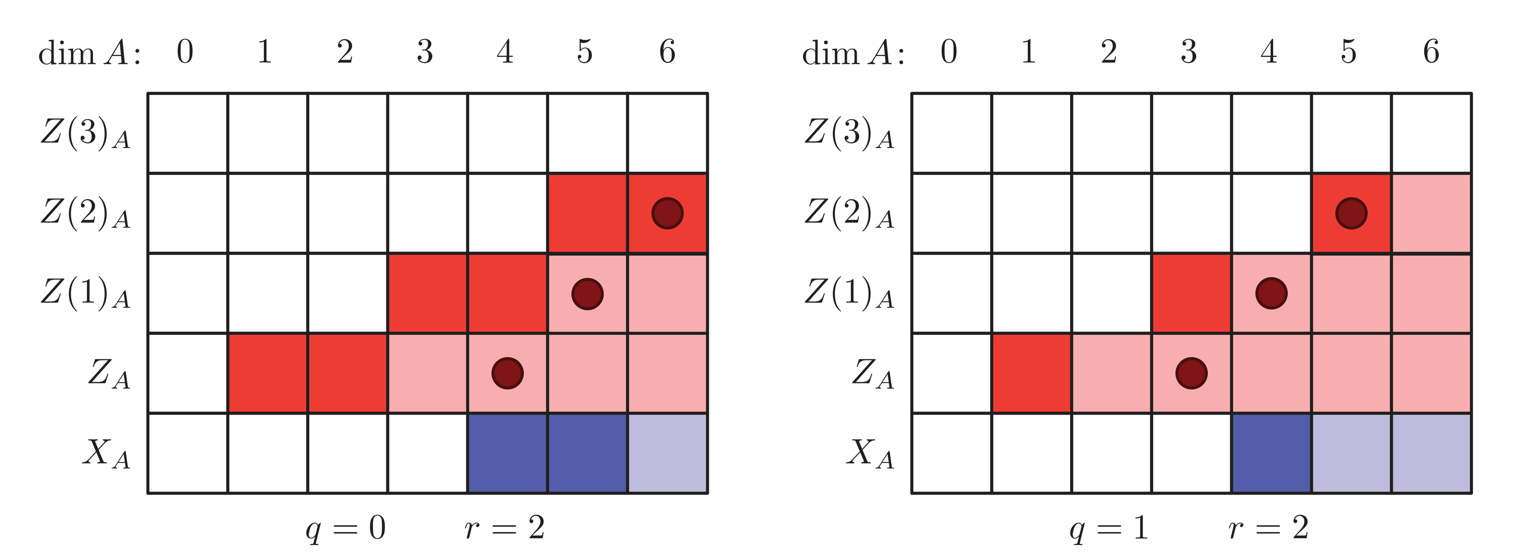
"Validity" Theorem

Fix a QRM code $QRM_m(q,r)$ with $q < r$.

For a given subcube operator $Z(k)_{x+\langle K\rangle}$, the nature of the logic implemented by $Z(k)_{x+\langle K\rangle}$ on $QRM_m(q,r)$ is determined by k and $|K|$:

Validity Theorem. $|K| \in$





"Logic" Theorem

Consider a standard subcube operators, i.e., $Z(k)_{\langle K\rangle}$ for $K \subseteq S$.

If $Z(k)_{\langle K\rangle}$ performs non-trivial logic then by the Validity Theorem, then

$$K \in \mathcal{Q}_k \equiv \{K' \subseteq \mathcal{Q} \mid q+kr+1 \le |K'| \le (k+1)r\}.$$

Definition. A set of logical qubits $\mathcal{J} \subseteq \mathcal{Q}$ is called a *minimal cover* for K if (1) $\bigcup_{J\in\mathcal{J}} J = K$ and (2) $|\mathcal{J}| = k+1$.

The collection of minimal covers for K is denoted by $\mathcal{F}(K)$.

Logic Theorem. If $K \in \mathcal{Q}_k$, then $Z(k)_{\langle K\rangle}$ implements:

$$Z(k)_{\langle K\rangle} \equiv \prod_{\mathcal{J}\in\mathcal{F}(K)} \overline{C^{\mathcal{J}}Z},$$

where $\overline{C^{\mathcal{J}}Z}$ is a $(k+1)$-qubit controlled-Z gate applied to the logical qubits in $\mathcal{J}$.

Notes:
(1) The result only holds if $q \ge 1$; we also characterize the $q=0$ case.
(2) We describe the circuits for arbitrary subcube operators, as well.

Ex: Logical circuit corresponding to the T operator applied to the $\langle 1,2,3,4,5\rangle$ subcube of $QRM_6(0,2)$.

Notes:

(1) The circuit is composed of $\overline{CCZ}$ gates.

(2) Pick a $\overline{CCZ}$ gate and take the union of the sets defining its qubits. The union will always be equal to $\{1,2,3,4,5\}$.

(3) Any logical qubit which has a 6 in its index set is unaffected.