Divide-and-conquer method for approximating output probabilities of geometrically-local, shallow-depth quantum circuits

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Problem statement

Let C be a geometrically-local shallow-depth quantum circuit acting on n qubits.

- **Goal:** Compute

$$\text{Pr}\Big[\text{measuring } x \text{ after preparing } C\left|0^{\otimes n}\right\rangle\Big] \pm \epsilon = \left|\langle x| C \left|0^{\otimes n}\right\rangle\right|^2 \pm \epsilon. \quad (1)$$

- What is the classical complexity of computing $|\langle x| C |0^{\otimes n}\rangle|^2$ to within additive error ϵ?
- In the *worst-case*, i.e. guaranteed run-time/error for all such circuits, and arbitrary 2-qubit gates.

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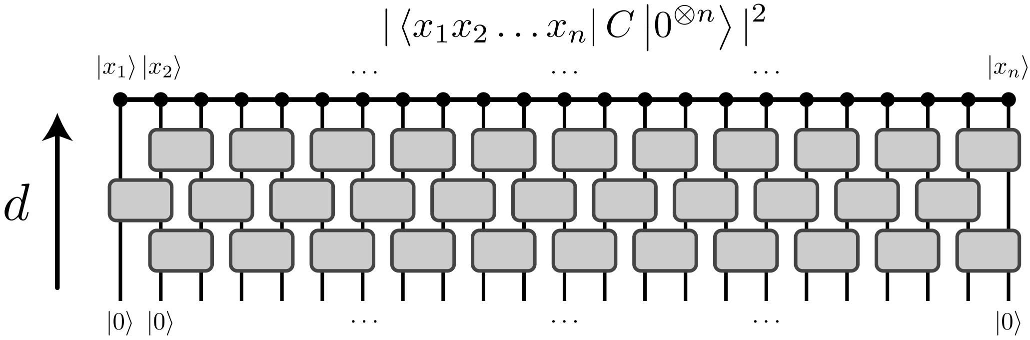
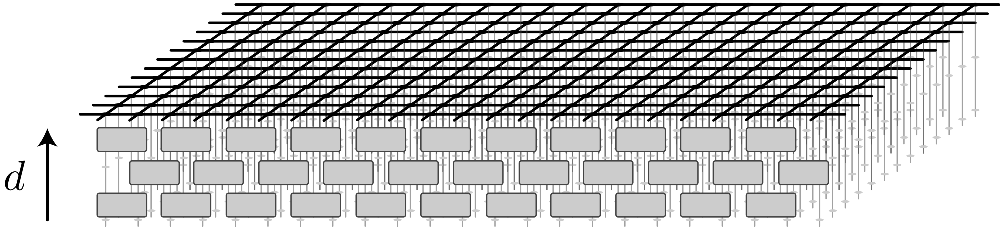
1D geometrically-local



- Notation: (1) input to circuits will be on the bottom, (2) outputs will be on top.
- Shallow-depth: $d = O(\log n)$.
- Classical matrix-product state algorithms are able to solve this problem efficiently and with inverse polynomial error.
- Note: Not restricted to brickwork architecture.

2D geometrically-local

$$|\langle x_1 x_2 \ldots x_n| C |0^{\otimes n}\rangle|^2$$

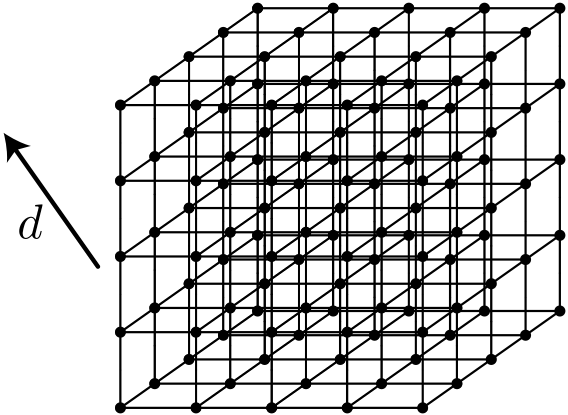


- Grid of $\sqrt{n} \times \sqrt{n}$ qubits.
- [BGM20] give a polynomial time classical algorithm to solve this case for inverse polynomial error. [1]
- Unclear how to extend the result of [BGM20] to 3 dimensions without an exponential blow-up.

[1] Sergey Bravyi, David Gosset, and Ramis Movassagh, arXiv:1909.11485

3D geometrically-local

$$|\langle x_1 x_2 \ldots x_n| C |0^{\otimes n}\rangle|^2$$



- Cube of $n^{1/3} \times n^{1/3} \times n^{1/3}$ qubits.
- Our result gives a quasi-polynomial time classical algorithm for this case.

Relation to sampling problems

- Sampling: output a bitstring $x \in \{0,1\}^n$ according to the probability distribution $p(x) = |\langle x| C |0^{\otimes n}\rangle|^2$.
- Estimating output probabilities: Compute $|\langle x| C |0^{\otimes n}\rangle|^2 \pm \epsilon$.

[2] Ramis Movassagh, arXiv:1909.06210

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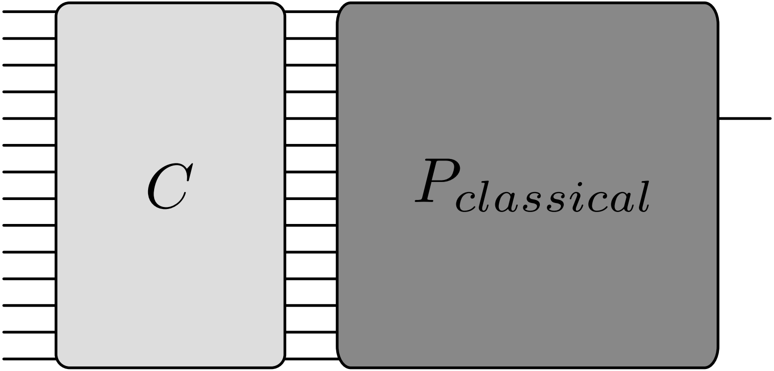
- [Mov20] In the worst case, #P-hard when $\epsilon \leq 2^{-n^2} \implies$ can only hope to solve this when $\epsilon \gg 2^{-n^2}$. [2]
- For random circuits $|\langle x| C |0^{\otimes n}\rangle|^2 \sim 2^{-n}$.
- In this work $|\langle x| C |0^{\otimes n}\rangle|^2 \geq 1/\text{poly}(n) \implies$ circuit must have some special properties.

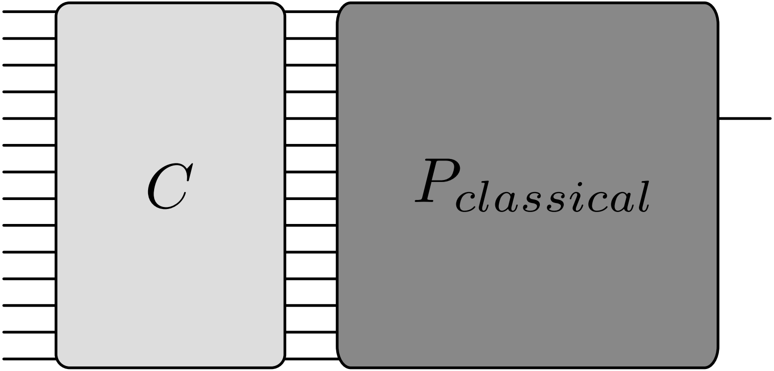
[2] Ramis Movassagh, arXiv:1909.06210

Why should we care about this error?



- Relevant for classically simulating some hybrid quantum algorithms.
- Geometrically-local quantum circuit combined with classical post-processing.

Why should we care about this error?



Such classical algorithms can simulate:

- $P_{classical} =$ AND: $|\langle 0^{\otimes n}| X^{\otimes n} C |0^{\otimes n}\rangle|^2 \pm \epsilon$, $P_{classical} =$ OR: $1 - |\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2 \pm \epsilon$.
- $P_{classical} =$ XOR: $\left|\langle 0^{\otimes n}| C Z^{\otimes n} C^\dagger |0^{\otimes n}\rangle\right|^2 \pm \epsilon$.

Note

- Only need to solve the problem for $|\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2 \pm \epsilon$ since

$$\left|\langle x| C \left|0^{\otimes n}\right\rangle\right|^2 = \left|\langle 0^{\otimes n}| (\bigotimes_i X^{x_i}) C \left|0^{\otimes n}\right\rangle\right|^2 \tag{2}$$

  and $(\bigotimes_i X_i)C$ is still geometrically-local, shallow-depth.
- In arXiv:2012.0546 we consider 3D geometrically-local, in this talk I will be discussing the methods in terms of 1D geometrically-local.

Note

- Only need to solve the problem for $|\langle 0^{\otimes n}| C |0^{\otimes n}\rangle|^2 \pm \epsilon$ since

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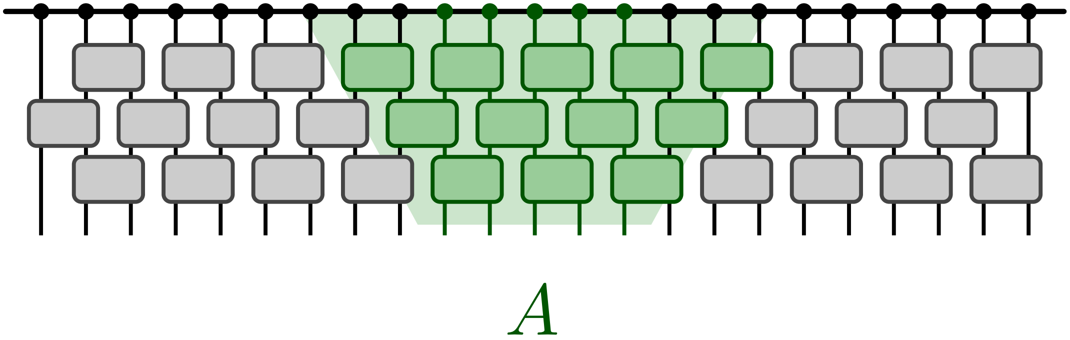
  and $(\bigotimes_i X_i)C$ is still geometrically-local, shallow-depth.
- In arXiv:2012.0546 we consider 3D geometrically-local, in this talk I will be discussing the methods in terms of 1D geometrically-local.
- **Notes:**
  - The algorithm has a divide-and-conquer structure.
  - Base-case will contain circuits in one fewer dimension.
  - Not explicitly giving the algorithm, just a single divide-and-conquer step.

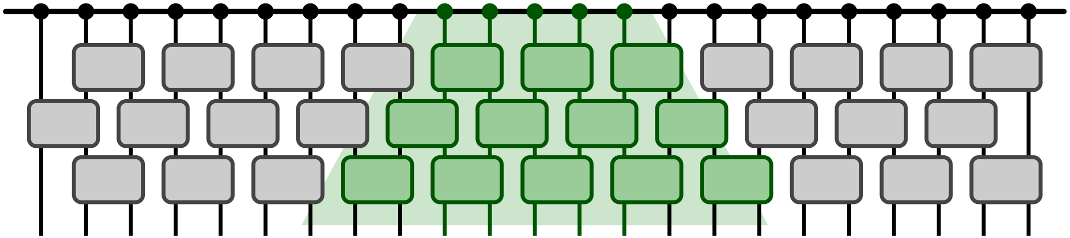
Preliminaries

Lightcone



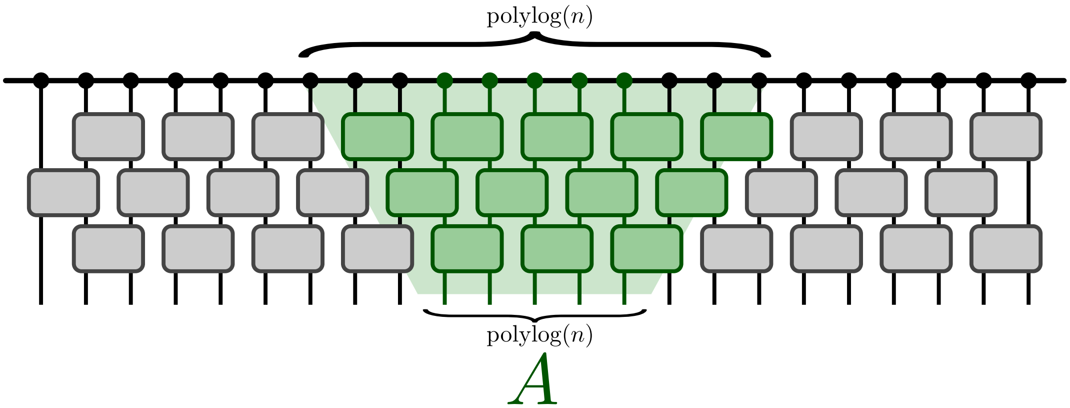
- Let A be a geometrically-local subset of the qubits.
- Lightcone of A will mean the gates in the lightcone or the qubits in the lightcone.

Reverse lightcone



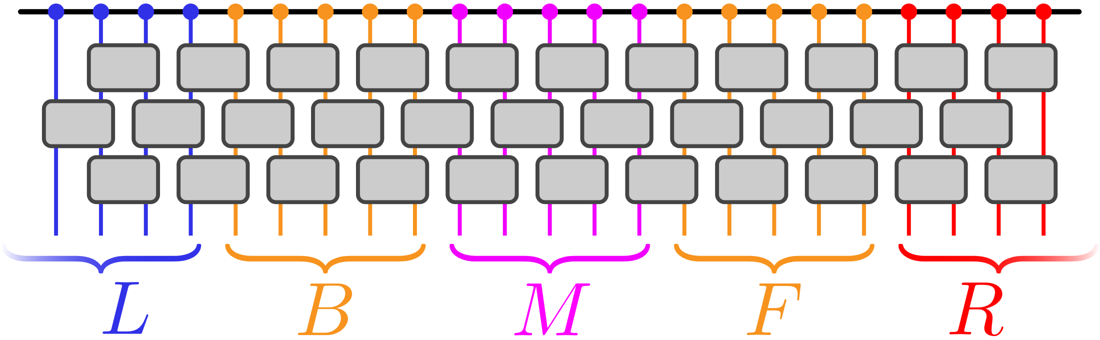
A

Motivation behind lightcones



If C is geometrically-local and shallow-depth, and $|A|$ is poly-logarithmic, then the [reverse] lightcone of A is only poly-logarithmically wide.

Slicing the circuit



A slice consists of three regions B, M, F. With appropriate widths, L and R can be lightcone-separated from M.

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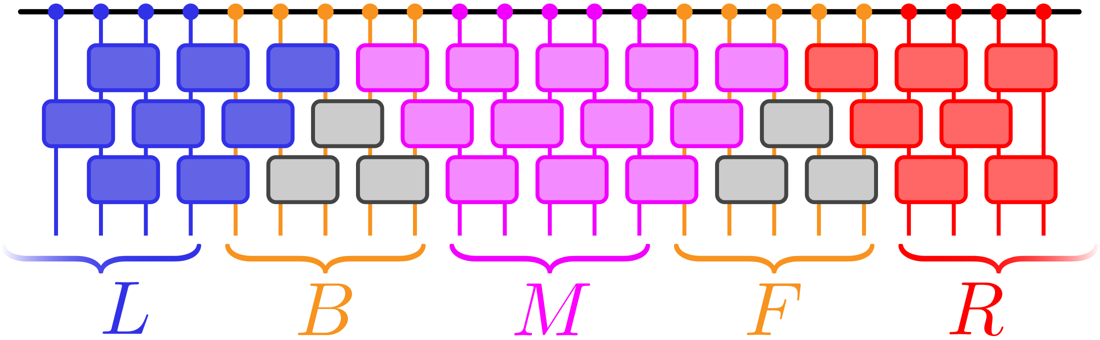

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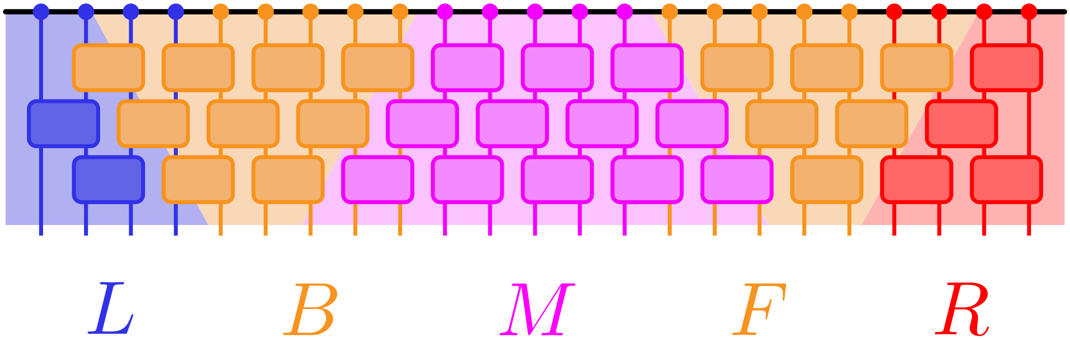
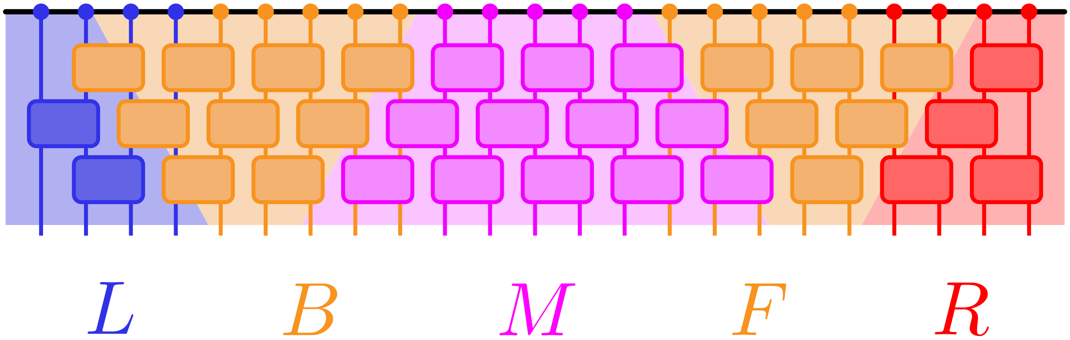


Subcircuits of C:

- $C_{B\cup M\cup F}$: gates in the reverse lightcone of M; C_L, C_R are the remaining gates.
- **Property:** $C = C_L \circ C_R \circ C_{B\cup M\cup F}$. The order is important!

Slicing the circuit



- C_{wrap}: gates in the lightcone of $B \cup M \cup F$ that are not in $C_{B \cup M \cup F}$.
- $C'_L \equiv C^{\dagger}_{L-Wrap} \circ C_L$.
- $C'_R \equiv C^{\dagger}_{R-Wrap} \circ C_R$.

Schmidt approximation

Definition

Let $|\psi\rangle_{B\cup F} \equiv \langle 0_M| C_{B\cup M\cup F} |0_{B\cup M\cup F}\rangle$.

- Note that, $\langle 0_{ALL}| C_{L\cup R} |0_{L\cup R}\rangle \otimes |\psi\rangle_{B\cup F} = \langle 0_{ALL}| C |0_{ALL}\rangle$.
- $|0_{ALL}\rangle$ will refer to the all zero state on the unmeasured qubits.

Schmidt approximation

Let $\left|\psi\right\rangle_{B\cup F} \approx \sum_i \lambda_i \left|v_i\right\rangle_B \otimes \left|w_i\right\rangle_F$ be the Schmidt decomposition of $\left|\psi\right\rangle_{B\cup F}$. Then

$$\left\langle 0_{ALL}\right| C_{L\cup R} \left|0_{L\cup R}\right\rangle \otimes \left|\psi\right\rangle_{B\cup F} \approx \sum_{i=1}^{p(n)} \lambda_i \left\langle 0_{ALL}\right| C_{L\cup R} \left|0_{L\cup R}\right\rangle \otimes \left|v_i\right\rangle_B \otimes \left|w_i\right\rangle_F \tag{3}$$

$$= \sum_{i=1}^{p(n)} \lambda_i \Big(\left\langle 0_{L\cup B}\right| C_L \left|0\right\rangle_L \otimes \left|v_i\right\rangle_B \Big)\Big(\left\langle 0_{F\cup R}\right| C_R \left|0_R\right\rangle \otimes \left|w_i\right\rangle_F \Big). \tag{4}$$

Schmidt approximation

Let $|\psi\rangle_{B\cup F} \approx \sum_i \lambda_i |v_i\rangle_B \otimes |w_i\rangle_F$ be the Schmidt decomposition of $|\psi\rangle_{B\cup F}$. Then

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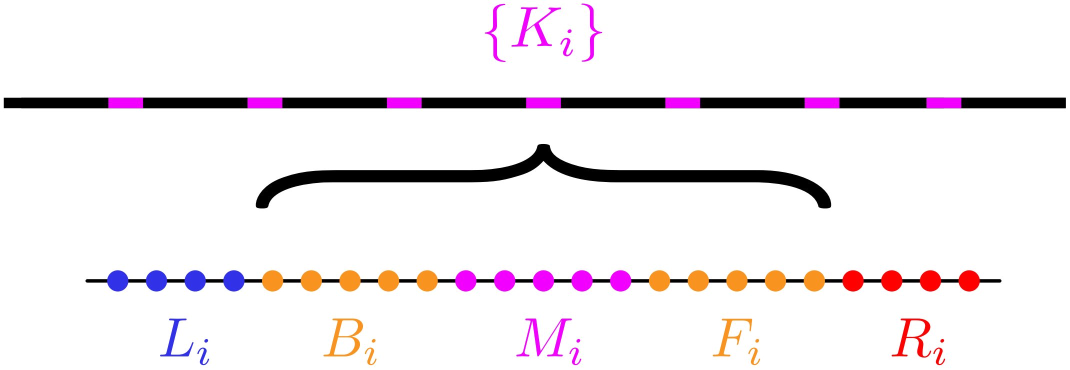
$$= \sum_{i=1}^{p(n)} \lambda_i \Big(\langle 0_{L\cup B}| C_L |0\rangle_L \otimes |v_i\rangle_B \Big)\Big(\langle 0_{F\cup R}| C_R |0_R\rangle \otimes |w_i\rangle_F \Big). \tag{4}$$

Notes:

- How could we construct $\lambda_i, |v_i\rangle_B, |w_i\rangle_F$ with geometrically-local, shallow-depth quantum circuits?
- Why should this state have most of its weight on a few Schmidt-coefficients?
- We do not solve the problem via the above equations.

Slicing the circuit

1. Let $\{K_i = B_i \cup M_i \cup F_i\}$ be a set of $\Omega(n)$ evenly spaced slices of width $O(d)$.



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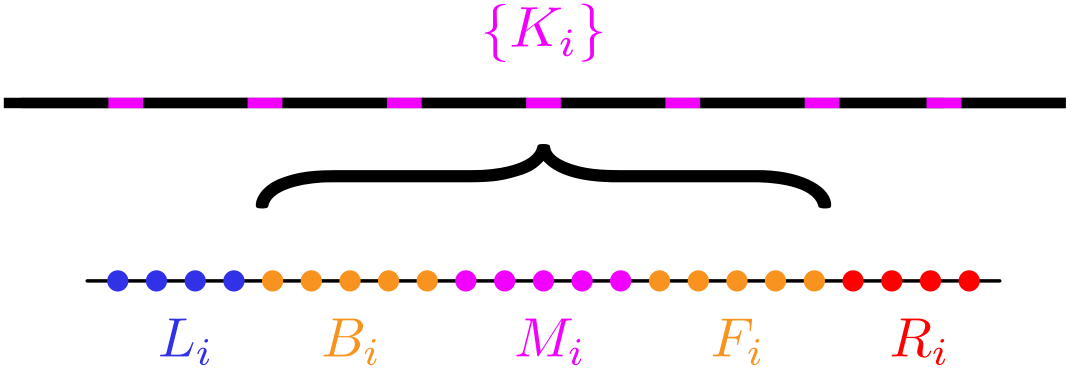
1. Let $\{K_i = B_i \cup M_i \cup F_i\}$ be a set of $\Omega(n)$ evenly spaced slices of width $O(d)$.
2. $C_{B_i \cup M_i \cup F_i}, C_{L,i}, C_{i,R}$, and C_{wrap_i} are all defined analogously as before.



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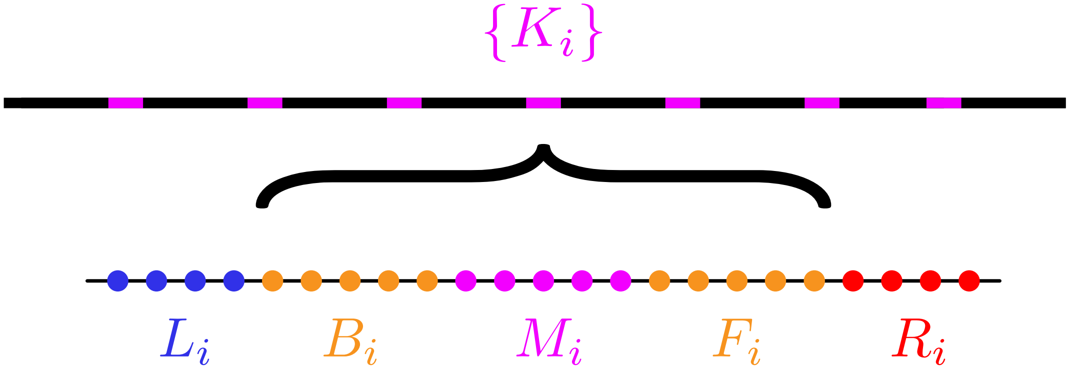

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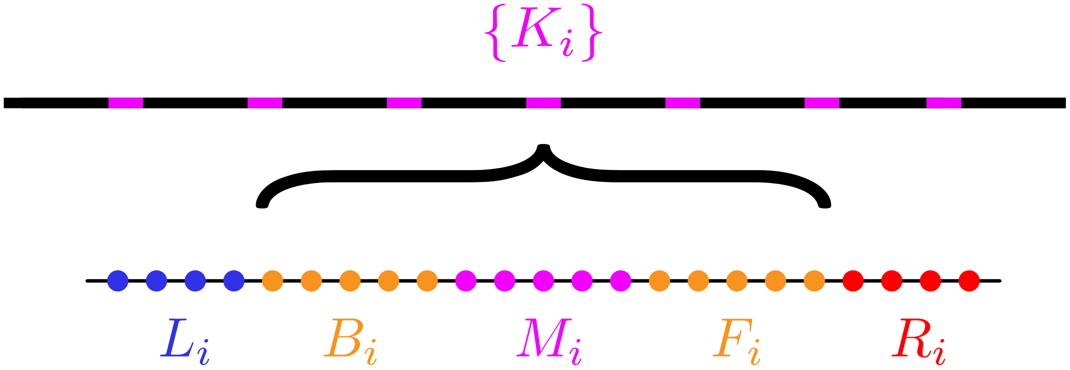
4. At a single slice: $C = C_{L,i} \circ C_{i,R} \circ C_{B_i \cup M_i \cup F_i}$.



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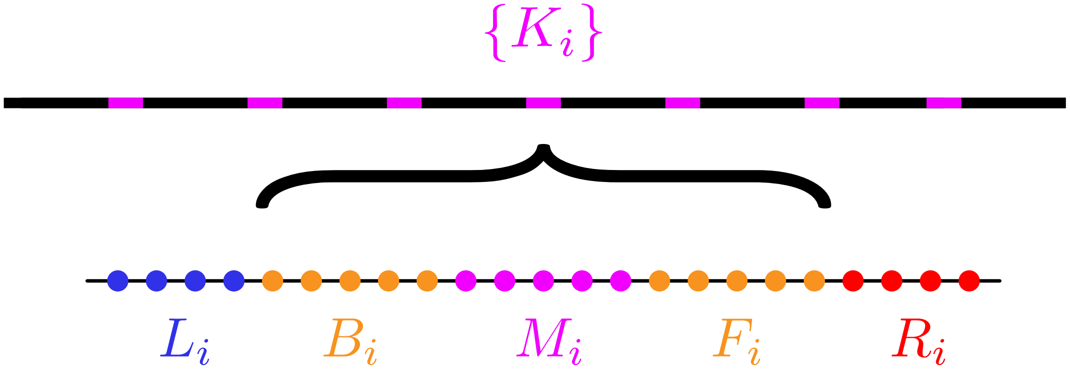
4. At a single slice: $C = C_{L,i} \circ C_{i,R} \circ C_{B_i \cup M_i \cup F_i}$.
5. At two slices: $C = C_{L,i} \circ C_{i,j} \circ C_{j,R} \circ C_{B_i \cup M_i \cup F_i} \circ C_{B_j \cup M_j \cup F_j}$



Heavy slices

Many of the slices have large leading Schmidt coefficients:

Lemma

Suppose $|\langle 0_{ALL}| C |0_{ALL}\rangle|^2 \geq 1/\text{poly}(n)$*. In every interval of length* $\log^7(n)$*, there are at least* $\log(n)$ *slices satisfying:*

$$\lambda_1 \geq 1 - \frac{1}{\log^4 n}. \tag{5}$$

- The set of slices that satisfying Equation (5) will be denoted as $K_{heavy} \subset \{K_i\}$.
- We will typically consider Δ heavy slices from an interval of width $\log^7(n)$.

Block-encoding

For slices in K_{heavy}, we can produce the projectors onto the leading Schmidt coefficients $(|w_1\rangle\langle w_1|_{F_i}, |v_1\rangle\langle v_1|_{B_i})$ via geometrically-local, shallow-depth quantum circuits.

Block-encoding

For slices in K_{heavy}, we can produce the projectors onto the leading Schmidt coefficients $(|w_1\rangle\langle w_1|_{F_i}, |v_1\rangle\langle v_1|_{B_i})$ via geometrically-local, shallow-depth quantum circuits.

Lemma (Lemma 53 of [GSLW19])

For any constant integer $K > 0$, the following is a geometrically-local quantum circuit which gives a block encoding for $\rho_F^K \equiv \mathrm{tr}_B(|\psi\rangle\langle\psi|_{B\cup F})^K$, and has depth $O(dK^2)$:

$$V_{\rho_F^k} = \prod_{i=1}^{k} \left(C^{\dagger}_{B_i \cup M_i' \cup F_i'} \otimes I_{M_i \cup F} \right) \cdot \left(I_{B_i} \otimes SWAP_{M_i \cup F, M_i' \cup F_i'} \right) \left(C_{B_i \cup M_i' \cup F_i'} \otimes I_{M_i \cup F} \right).$$

In other words,

$$\rho_F^K = \left(\langle 0|_{\mathcal{B}_k \cup \mathcal{M}_k' \cup \mathcal{F}_k' \cup \mathcal{M}_k} \otimes I_F \right) V_{\rho_F^k} \left(|0\rangle_{\mathcal{B}_k \cup \mathcal{M}_k' \cup \mathcal{F}_k' \cup \mathcal{M}_k} \otimes I_F \right)$$

where $\mathcal{B}_k = B_1 \cup B_2 \cup \cdots \cup B_k$, $\mathcal{M}_k' = M_1' \cup M_2' \cup \cdots \cup M_k'$, etc.

Takeaway: $V_{\rho_F^k}$ is a geometrically-local, shallow-depth circuit which produces ρ_F^K after post-selection.

Projector lemma

Define:

$$P_{F_i}^K \equiv \frac{1}{\lambda_1^K} \left\langle 0^{B_i,M_i,F_i^1,\ldots F_i^k} \right| V_{P_{F_i}^k} \left| 0^{B_i,M_i,F_i^1,\ldots F_i^k} \right\rangle \tag{6}$$

and

$$P_{B_i}^K \equiv \frac{1}{\lambda_1^K} \left\langle 0^{F_i,M_i,B_i^1,\ldots B_i^K} \right| V_{P_{B_i}^k} \left| 0^{F_i,M_i,B_i^1,\ldots B_i^K} \right\rangle \tag{7}$$

Lemma

For any $K_i \in K_{heavy}$,

$$\|P_{F_i}^K - |w_1\rangle \langle w_1|_{F_i} \|_1 \leq \frac{1}{\text{poly}(n)} \tag{8}$$

$$\textit{and } \|P_{B_i}^K - |v_1\rangle \langle v_1|_{B_i} \|_1 \leq \frac{1}{\text{poly}(n)} \tag{9}$$

Note: In this case, K is the number of times the block-encoded circuit is going to be applied. The effect of K is hidden in this lemma.

Projector lemma

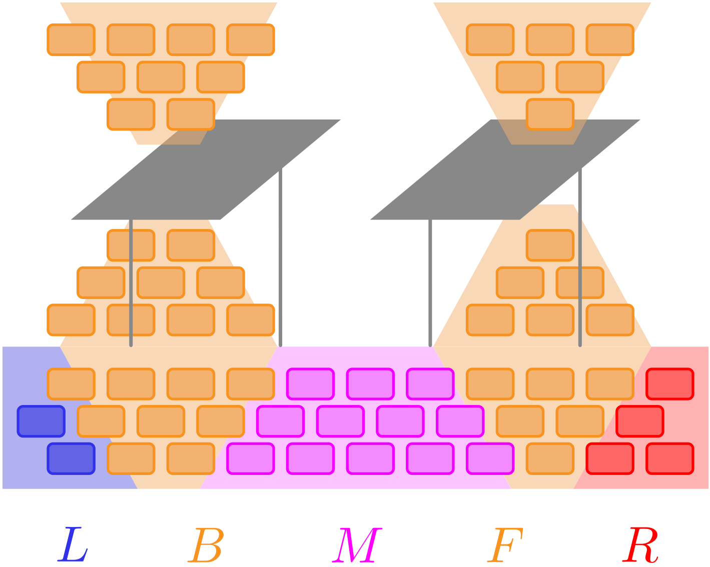
Definition

$$\Pi^K_{F_i} \equiv C_{Wrap_i} P^K_{F_i} C^\dagger_{Wrap_i} \tag{10}$$

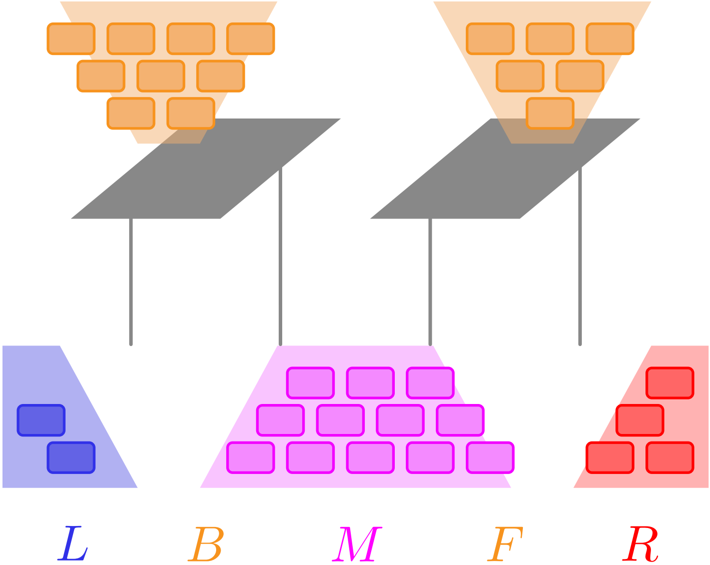
For a single i, consider

$$\Pi^K_{F_i} C = \Pi^K_{F_i} (C_{L,i} \circ C_{i,R} \circ C_{B_i \cup M_i \cup F_i}). \tag{11}$$

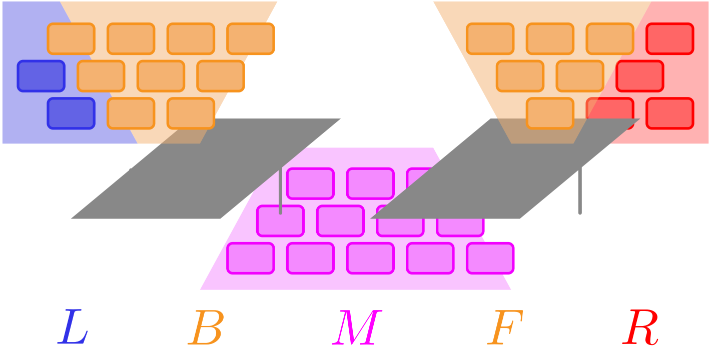
What does this do?



$$\Pi_{F_i}^{K}(C_{L,i} \circ C_{i,R} \circ C_{B_i \cup M_i \cup F_i}) = C_{Wrap_i} P_{F_i}^{K} C_{Wrap_i}^{\dagger} (C_{L,i} \circ C_{i,R} \circ C_{B_i \cup M_i \cup F_i})$$



$$= C_{Wrap_i}(C'_{L,i} \circ C'_{i,R} \circ P^K_{F_i} C_{B_i \cup M_i \cup F_i})$$



$$\Pi^K_{F_i} C = C_{L,i} \circ C_{i,R} \circ P^K_{F_i} C_{B_i \cup M_i \cup F_i}$$

Linear combination lemma

Linear Combination Lemma

Definition

Let $\sigma \in \mathcal{P}([\Delta]) \setminus \emptyset$ where $[\Delta] = \{1, \ldots, \Delta\}$. Define

$$|\Psi_\sigma\rangle = \bigotimes_{j \in \sigma} \Pi_{F_j}^{K} \bigotimes_{i \in [\Delta]} \langle 0_{M_i}| \, C \, |0_{ALL}\rangle . \quad (12)$$

Linear Combination Lemma

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$$|\Psi_\sigma\rangle = \bigotimes_{j \in \sigma} \Pi^K_{F_j} \bigotimes_{i \in [\Delta]} \langle 0_{M_i}| C |0_{ALL}\rangle . \tag{12}$$

Lemma

Consider a set K_{heavy} of heavy slices. Then, for any subset of Δ slices, $\{K_i\}_{i\in[\Delta]} \subseteq K_{heavy}$:

$$\left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} |\Psi_\sigma\rangle \langle \Psi_\sigma| \right\| = \left\| |\Psi_\emptyset\rangle \langle \Psi_\emptyset| - \sum_{\sigma \in \mathcal{P}([\Delta]) \setminus \emptyset} (-1)^{|\sigma|+1} |\Psi_\sigma\rangle \langle \Psi_\sigma| \right\| \leq 1/\text{poly}(n) \tag{13}$$

Note: $\langle 0_{ALL}|\Psi_\emptyset\rangle \langle \Psi_\emptyset|0_{ALL}\rangle = \langle 0_{ALL}| \bigotimes_{i\in[\Delta]} \langle 0_{M_i}| C |0_{ALL}\rangle \langle 0_{ALL}| C^\dagger \bigotimes_{i\in[\Delta]} \langle 0_{M_i}|0_{ALL}\rangle = |\langle 0_{ALL}| C |0_{ALL}\rangle|^2$

Proof

We will use the notation $\rho[A] = A\left|0_{ALL}\right\rangle\left\langle 0_{ALL}\right| A^\dagger$. Note that $\rho[BA] = B\rho[A]B^\dagger$. With this

$$\left\| \sum_{\sigma\in\mathcal{P}([\Delta])} (-1)^{|\sigma|} \left|\Psi_\sigma\right\rangle\left\langle\Psi_\sigma\right| \right\| = \left\| \sum_{\sigma\in\mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho\left[\bigotimes_{j\in\sigma} \Pi_{F_j}^K \bigotimes_{i\in[\Delta]} \left\langle 0_{M_i}\right| C \right] \right\|. \tag{14}$$

We first consider the above without post-selection

$$\left\| \sum_{\sigma\in\mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho\left[\bigotimes_{j\in\sigma} \Pi_{F_j}^K C \right] \right\|. \tag{15}$$

Proof

Recall:

$$\text{for } \sigma = \{i\}: \quad C = C_{L,j} \circ C_{j,R} \circ C_{B_j \cup M_j \cup F_j} \tag{16}$$

$$\text{for } \sigma = \{i,j\}: \quad C = C_{L,i} \circ C_{i,j} \circ C_{j,R} \circ C_{B_i \cup M_i \cup F_i} \circ C_{B_j \cup M_j \cup F_j} \tag{17}$$

$$\left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho \left[\bigotimes_{j \in \sigma} \Pi^K_{F_j} C \right] \right\| = \tag{18}$$

$$\left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho \left[\bigotimes_{j \in \sigma} \Pi^K_{F_j} \left(C_{L,\sigma_1} \circ \bigotimes_{j \in [|\sigma|-1]} C_{\sigma_j, \sigma_{j+1}} \circ C_{\sigma_{|\sigma|}, R} \circ \bigotimes_{j \in \sigma} C_{B_j \cup M_j \cup F_j} \right) \right] \right\| \tag{19}$$

Proof

For a single $i \in \sigma \subseteq [\Delta]$, recall the following:

$$\Pi^K_{F_i}(C_{L,i} \circ C_{i,R} \circ C_{B_i \cup M_i \cup F_i}) = C_{L,i} \circ C_{i,R} \circ P^K_{F_i} C_{B_i \cup M_i \cup F_i} \tag{20}$$

With this, we have

$$\left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho \left[\bigotimes_{j \in \sigma} \Pi^K_{F_j} \left(C_{L,\sigma_1} \circ \bigotimes_{j \in [|\sigma|-1]} C_{\sigma_j, \sigma_{j+1}} \circ C_{\sigma_{|\sigma|}, R} \circ \bigotimes_{j \in \sigma} C_{B_j \cup M_j \cup F_j} \right) \right] \right\| = \tag{21}$$

$$\left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho \left[C_{L,\sigma_1} \circ \bigotimes_{j \in [|\sigma|-1]} C_{\sigma_j, \sigma_{j+1}} \circ C_{\sigma_{|\sigma|}, R} \circ \bigotimes_{j \in \sigma} P^K_{F_j} C_{B_j \cup M_j \cup F_j} \right] \right\| \tag{22}$$

Proof

Consider if $\Delta = 2$. Our previous equation becomes:

$$\left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho \left[C_{L,\sigma_1} \circ \bigotimes_{j \in [|\sigma|-1]} C_{\sigma_j, \sigma_{j+1}} \circ C_{\sigma_{|\sigma|}, R} \circ \bigotimes_{j \in \sigma} P^K_{F_j} C_{B_j \cup M_j \cup F_j} \right] \right\| = \tag{23}$$

$$\Big\| \rho[C] \tag{24}$$

$$-\rho \left[C_{L,1} \circ C_{1,R} \circ P^K_{F_1} C_{B_1 \cup M_1 \cup F_1} \right) \tag{25}$$

$$-\rho \left[C_{L,2} \circ C_{2,R} \circ P^K_{F_2} C_{B_2 \cup M_2 \cup F_2} \right] \tag{26}$$

$$+\rho \left[C_{L,1} \circ C_{1,2} \circ C_{2,R} \circ P^K_{F_1} C_{B_1 \cup M_1 \cup F_1} \circ P^K_{F_2} C_{B_2 \cup M_2 \cup F_2} \right] \Big\| \tag{27}$$

$$\tag{28}$$

Proof

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$$\left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho \left[C_{L,\sigma_1} \circ \bigotimes_{j \in [|\sigma|-1]} C_{\sigma_j,\sigma_{j+1}} \circ C_{\sigma_{|\sigma|},R} \circ \bigotimes_{j \in \sigma} P^K_{F_j} C_{B_j \cup M_j \cup F_j} \right] \right\| = \quad (23)$$

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$$\quad (28)$$

Expanding the circuit in each:

$$\Big\| \rho[C_{L,1} \circ C_{1,2} \circ C_{2,R} \circ C_{B_1 \cup M_1 \cup F_1} \circ C_{B_2 \cup M_2 \cup F_2}] \quad (29)$$

$$-\rho \left[C_{L,1} \circ C_{1,2} \circ C_{2,R} \circ P^K_{F_1} C_{B_1 \cup M_1 \cup F_1} \circ C_{B_2 \cup M_2 \cup F_2} \right] \quad (30)$$

$$-\rho \left[C_{L,1} \circ C_{1,2} \circ C_{2,R} \circ C_{B_1 \cup M_1 \cup F_1} \circ P^K_{F_2} C_{B_2 \cup M_2 \cup F_2} \right] \quad (31)$$

$$+\rho \left[C_{L,1} \circ C_{1,2} \circ C_{2,R} \circ P^K_{F_1} C_{B_1 \cup M_1 \cup F_1} \circ P^K_{F_2} C_{B_2 \cup M_2 \cup F_2} \right] \Big\| \quad (32)$$

Proof

$$\Big\|\Big(C_{L,1} \circ C_{1,2} \circ C_{2,R}\Big) \circ \Big(\rho[C_{B_1\cup M_1\cup F_1} \circ C_{B_2\cup M_2\cup F_2}] - \rho\left[P^K_{F_1} C_{B_1\cup M_1\cup F_1} \circ C_{B_2\cup M_2\cup F_2}\right]$$
$$-\rho\left[C_{B_1\cup M_1\cup F_1} \circ P^K_{F_2} C_{B_2\cup M_2\cup F_2}\right] + \rho\left[P^K_{F_1} C_{B_1\cup M_1\cup F_1} \circ P^K_{F_2} C_{B_2\cup M_2\cup F_2}\right]\Big) \circ \Big(C_{L,1} \circ C_{1,2} \circ C_{2,R}\Big)^\dagger\Big\|$$

Proof

$$\Big\|\Big(C_{L,1} \circ C_{1,2} \circ C_{2,R}\Big) \circ \Big(\rho[C_{B_1 \cup M_1 \cup F_1} \circ C_{B_2 \cup M_2 \cup F_2}] - \rho\left[P^K_{F_1} C_{B_1 \cup M_1 \cup F_1} \circ C_{B_2 \cup M_2 \cup F_2}\right]$$

$$-\rho\left[C_{B_1 \cup M_1 \cup F_1} \circ P^K_{F_2} C_{B_2 \cup M_2 \cup F_2}\right] + \rho\left[P^K_{F_1} C_{B_1 \cup M_1 \cup F_1} \circ P^K_{F_2} C_{B_2 \cup M_2 \cup F_2}\right]\Big) \circ \Big(C_{L,1} \circ C_{1,2} \circ C_{2,R}\Big)^\dagger\Big\|$$

Can rewrite as a tensor product:

$$\Big\|\Big(C_{L,1} \circ C_{1,2} \circ C_{2,R}\Big) \circ \bigotimes_{j \in \{1,2\}} \Big(\rho[C_{B_j \cup M_j \cup F_j}] - \rho[P^K_{F_j} C_{B_j \cup M_j \cup F_j}]\Big) \circ \Big(C_{L,1} \circ C_{1,2} \circ C_{2,R}\Big)^\dagger\Big\| \quad (33)$$

Proof

In general, we have:

$$= \left\| \left(C_{L,1} \circ \bigotimes_{j \in [\Delta-1]} C_{\sigma_j,\sigma_{j+1}} \circ C_{\sigma_\Delta,R} \right) \circ \bigotimes_{j \in [\Delta]} \left(\rho[C_{B_j \cup M_j \cup F_j}] - \rho[P^K_{F_j} C_{B_j \cup M_j \cup F_j}] \right) \right. \tag{34}$$

$$\left. \circ \left(C_{L,1} \circ \bigotimes_{j \in [\Delta-1]} C_{\sigma_j,\sigma_{j+1}} \circ C_{\sigma_\Delta,R} \right)^\dagger \right\|. \tag{35}$$

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Since $(C_{L,1} \circ \bigotimes_{j \in [\Delta-1]} C_{\sigma_j,\sigma_{j+1}} \circ C_{\sigma_\Delta,R})$ is unitary:

$$= \prod_{j \in \Delta} \left\| \rho[C_{B_j \cup M_j \cup F_j}] - \rho[P^K_{F_j} C_{B_j \cup M_j \cup F_j}] \right\| \tag{36}$$

Proof

To sum-up to this point:

$$\left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho \left[\bigotimes_{j \in \sigma} \Pi_{F_j}^{K} C \right] \right\| = \prod_{j \in \Delta} \left\| \rho[C_{B_j \cup M_j \cup F_j}] - \rho[P_{F_j}^{K} C_{B_j \cup M_j \cup F_j}] \right\| . \tag{37}$$

Proof

Since $P^K_{F_j}$, $\Pi^K_{F_j}$ act trivially on M_j, the previous work holds even under post-selection on M_j:

$$\left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho \left[(\otimes_{j\in\sigma} \Pi^K_{F_j} \langle 0_{M_j} |) C \right] \right\| = \prod_{j\in[\Delta]} \left\| \rho \left[\langle 0_{M_j} | \, C_{B_j \cup M_j \cup F_j} \right] - \rho \left[P^K_{F_j} \langle 0_{M_j} | \, C_{B_j \cup M_j \cup F_j} \right] \right\| \tag{38}$$

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Each term in the above product can be bounded as:

$$\left\| \rho \left[\langle 0_{M_j} | \, C_{B_j \cup M_j \cup F_j} \right] - \rho \left[P^K_{F_j} \langle 0_{M_j} | \, C_{B_j \cup M_j \cup F_j} \right] \right\| \leq \frac{1}{\log^4 n} \tag{39}$$

Proof

Ultimately,

$$\left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} \left|\Psi_\sigma\right\rangle \left\langle\Psi_\sigma\right| \right\| = \left\| \sum_{\sigma \in \mathcal{P}([\Delta])} (-1)^{|\sigma|} \rho \left[\bigotimes_{j \in \sigma} \Pi_{F_j}^K \bigotimes_{i \in [\Delta]} \langle 0_{M_i} | \, C \right] \right\| \tag{40}$$

$$= \prod_{j \in [\Delta]} \left\| \rho \left[\langle 0_{M_j} | \, C_{B_j \cup M_j \cup F_j} \right] - \rho \left[P_{F_j}^K \langle 0_{M_j} | \, C_{B_j \cup M_j \cup F_j} \right] \right\| \tag{41}$$

$$\leq \left(\frac{1}{\log^4 n}\right)^\Delta \leq 1/\mathrm{poly}(n), \tag{42}$$

when $\Delta = \log n$. ∎

Note: we can find $\log n$ slices in each region of size $\log^7 n$.

Approximations

We have:

$$\left\| |\langle 0_{ALL}| C |0_{ALL}\rangle|^2 - \sum_{\sigma \in \mathcal{P}([\Delta])\setminus\emptyset} (-1)^{|\sigma|+1} \langle 0_{ALL}|\Psi_\sigma\rangle \langle \Psi_\sigma|0_{ALL}\rangle \right\| \leq 1/\text{poly}(n), \tag{43}$$

But, $|\Psi_\sigma\rangle = \bigotimes_{j\in\sigma} \Pi_{F_j}^K \bigotimes_{i\in[\Delta]} \langle 0_{M_i}| C |0_{ALL}\rangle$ are produced by circuits acting on all n qubits.

Approximations

Approximations

Recall: $|\Psi_\sigma\rangle = \bigotimes_{j\in\sigma} \Pi_{F_j}^K \bigotimes_{i\in[\Delta]} \langle 0_{M_i}| C |0_{ALL}\rangle$.

Lemma

Each $|\Omega_j\rangle = \Pi_{F_j}^K \langle 0_{M_i}| C |0_{ALL}\rangle$ *is close to a product state across the cut* $B_j \cup M_j \cup F_j$.

$$\left\| |\Omega_j\rangle \langle \Omega_j| - 1/\lambda_1^j \sigma_{L_j} \otimes \sigma_{R_j} \right\| \leq \frac{1}{\log^4 n} \tag{44}$$

The state σ_{L_j} (resp. σ_{R_j}) is defined using $C_{B_j\cup M_j \cup F_j}$, $P_{F_j}^K$ (resp. $P_{B_j}^K$), and $C_{L,j}$ (resp. $C_{j,R}$).

Approximations

Given 1D geometrically-local, depth-d quantum circuit C on n qubits, we wish to approximate $|\langle 0_{ALL}| C |0_{ALL}\rangle|^2$ via

$$\sum_{\sigma \in \mathcal{P}([\Delta]) \setminus \emptyset} (-1)^{|\sigma|+1} \langle 0_{ALL}|\Psi_\sigma\rangle \langle \Psi_\sigma|0_{ALL}\rangle . \tag{45}$$

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- The remaining $O(\Delta^2)$ act on at most $\log^7 n$ qubits.
- The quantities $|\langle 0_{ALL}| \Gamma_\ell |0_{ALL}\rangle|^2$ can be used to approximate Equation (45).

Extending to 3D circuits

3D Circuits

C is a 3D geometrically-local shallow quantum circuit on $n^{1/3} \times n^{1/3} \times n^{1/3}$ qubits. We will perform division along a single dimension of the cube. Choose $\Omega(n^{1/3})$ slices from along this dimension.

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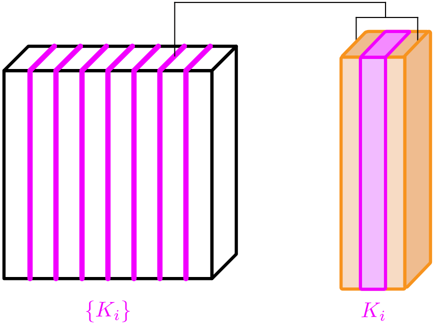

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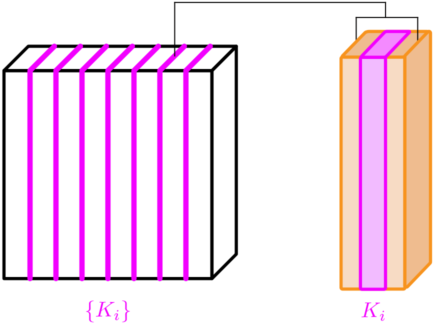

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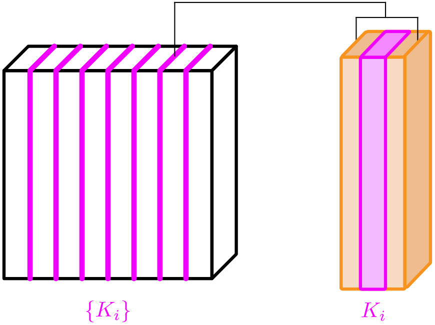

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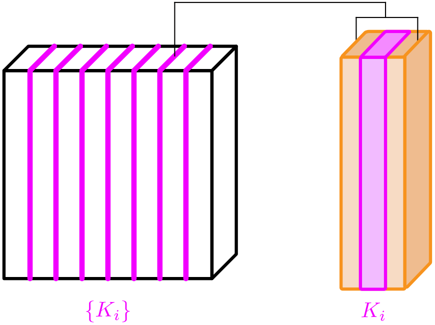
4. Use the algorithm from [BGM20] to approximate $|\langle 0_{ALL}| \Gamma_\ell |0_{ALL}\rangle|^2$ for the remaining circuits.



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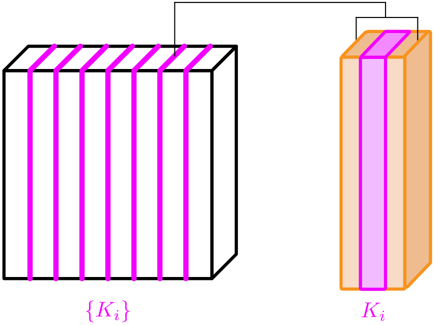
5. Combine solutions to approximate $|\langle 0_{ALL}| C |0_{ALL}\rangle|^2$.



3D Circuits

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3D Circuits

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- We saw each of these parameters show up in the approximation error (approximating the original quantity with a weighted sum of new problems, approximating these new problems with smaller problems). They also show up in the run-time analysis.
- Parameters can be chosen so that run-time is quasi-polynomial:

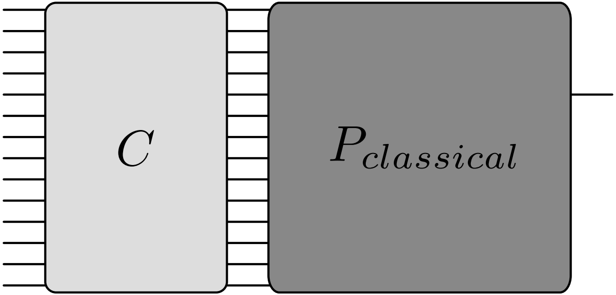
$$T(n) = 2^{d^3 \cdot \text{polylog}(n)}. \tag{46}$$

and error is inverse polynomial, $f(n) \leq 1/\text{poly}(n)$.

Recursive run-time and error analyses/full algorithm description can be found in the paper.

Open problems

- Improve to polynomial run-time.
- Recursively approximate output probabilities of any D-dimensional geometrically-local circuit.
- Consider circuits that are low-depth but not necessarily geometrically local.
- Estimate output bit of low-depth quantum circuit combined with classical post-processing.



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