Hamiltonians whose low-energy states require $\Omega(n)$ T gates

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Abstract

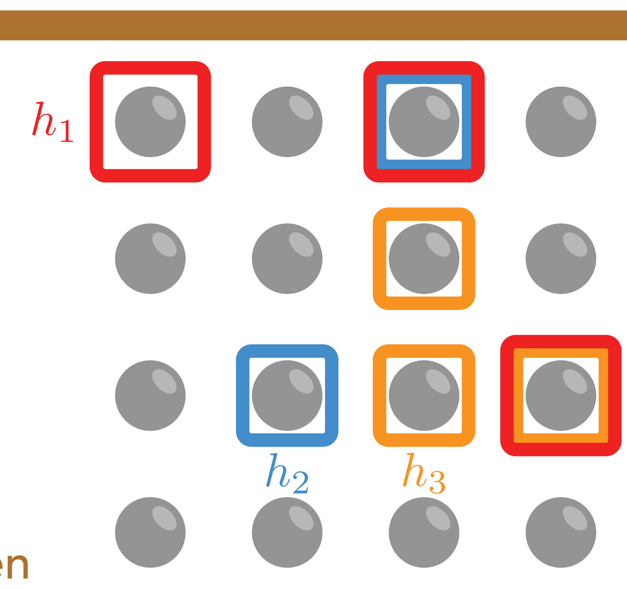
- The QMA-hardness of the Local Hamiltonian (LH) problem with $1/\mathrm{poly}(n)$ gap implies that we cannot $1/\mathrm{poly}(n)$-approximate the ground-state energy of arbitrary local Hamiltonians in BQP, unless BQP=QMA.

- The **Quantum PCP Conjecture** (QPCP) asserts that the same is true even for a contant, $\Omega(1)$, relative promise gap. If true, QPCP implies the existence of Hamiltonians with interesting low-energy space properties.

- **We exhibit one such prerequisite to QPCP:** an explicit local Hamiltonian whose low-energy states all require $\omega(\log n)$ T gates, i.e. they are very non-stabilizer. In fact, we show a stronger result that the low-energy states require $\Omega(n)$ T gates, which is not necessarily implied by QPCP.

- Applying our procedure to the NLTS family from [ABN22] **yields an NLTS Hamiltonian whose low-energy states also require $\Omega(n)$ T gates.**

"QPCP Hamiltonians"



Local Hamiltonian: $H = \frac{1}{m}\sum_{i=1}^{m} h_i \otimes I_{2^{n-k}}$

Ground-state energy: $E_{gs} \equiv \min_{|\psi\rangle} \langle\psi| H |\psi\rangle$

LH Problem: given H, ε>0, δ(n), decide between

(1) $E_{gs} \leq \varepsilon$ (2) $E_{gs} \geq \varepsilon + \delta(n)$

If $\mathcal{C} \subsetneq$ QMA = QPCP is conjectured, then for some constant δ>0 we cannot estimate $E_{gs} \pm \delta$ in $\mathcal{C}$.

$\Longrightarrow$ **No low-energy state of H can admit energy estimation in $\mathcal{C}$.**

Types of States

Stabilizer group: $S(|\psi\rangle) \equiv \{P \in \mathcal{P}_n \mid P|\psi\rangle = |\psi\rangle\}$

Stabilizer state: prepared by only Clifford gates $\Leftrightarrow |S(|\psi\rangle)| = 2^n$

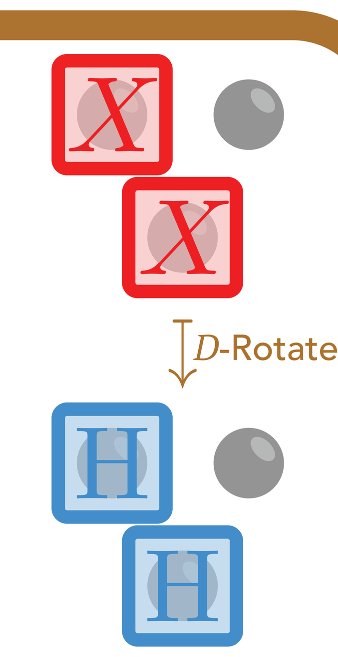
Size of $S(|\psi\rangle)$ vs. T-count: prepared by ≤ t T gates $\Rightarrow |S(|\psi\rangle)| \geq 2^{n-t}$

Almost-Clifford state: prepared by ≤ $\log n$ T gates

States and Hamiltonians

State	Energy Estimation Algorithm	Hamiltonian Implication
Trivial (low-depth circuit)	NP via a light-cone argument	NLTS [ABN22]
"Sampleable"	MA via dequantizing QSVT [GL22]	NLSS [GL22] (open)
Stabilizer	NP via stabilizer generators	NLCS [CCNN23]
Almost-Clifford	**NP via combo of light-cone + stabilizer generators**	**NLACS [this work]**
…	…	…

Rotating CSS Hamiltonians



CSS Hamiltonian: $H = \frac{1}{m}\sum_{i=1}^{m_x} \frac{1}{2}(I - X^{\otimes k})_i + \frac{1}{m}\sum_{i=1}^{m_z} \frac{1}{2}(I - Z^{\otimes k})_i$

Ground-states of a CSS Hamiltonian are highly-stabilizer.
Key idea: rotate to a basis which is far from stabilizer.

Single-qubit gates: H, Hadamard, and $D \equiv e^{i\frac{\pi}{8}Y}$

Rotated CSS Hamiltonian: $\tilde{H} \equiv D^{\otimes n} H D^{\dagger\otimes n} = \frac{1}{m}\sum_{i=1}^{m_x} \frac{1}{2}(I - \mathrm{H}^{\otimes k})_i + \ldots$

Theorem. There are ε>0 and 0<c<1 s.t. if $|\psi\rangle$ can be prepared by ≤ cn T gates, then $|\psi\rangle$ has energy $\langle\psi|\tilde{H}|\psi\rangle \geq \epsilon$.

Corollary 1. For every 0<ε<sin²(π/8), ε-low-energy states of $\frac{1}{n}\sum_{i=1}^{n} D\,|-\rangle\langle-|_i\,D^\dagger$ require $\Omega(n)$ T gates to prepare.

Corollary 2. For the D-rotated NLTS family from [ABN22], all states of low-enough constant energy require $\Omega(n)$ T gates and $\Omega(\log n)$ depth to prepare.

Local Energy Bound

Consider a local term, $h = \frac{1}{2}(I - \mathrm{H}^{\otimes k})$, and a state $|\psi\rangle$ with $S(|\psi\rangle) \equiv G$, prepared by ≤ cn T gates.

Step 1: If G "looks like" a full stabilizer group at h, then a local energy bound holds!

$$\langle\psi| h |\psi\rangle \geq \sin^2(\frac{\pi}{8})$$

Locally-commuting at h: ignoring the Paulis outside of supp(h), all terms commute.

$I\ I\ X\ I$
$I\ X\ Z\ Y$
$X\ I\ Z\ X$
$X\ X\ Y\ I$

Consider these four Pauli operators. The "local views" of every operator on qubits 1 and 2 mutually commute, even though the four operators mutually anti-commute.

Pseudo-stabilizer state at h: G has a subgroup, G_h, which is locally-commuting at h and has size 2^k. (Largest possible size)

A Pauli group which locally-commutes on qubits 1 and 2, and has maximal possible size 2²=4.

$I\ I\ I\ I$
$I\ X\ I\ Y$
$X\ I\ Z\ I$
$X\ X\ Z\ Y$

If G contains these four operators, then

$$\langle\psi| \frac{1}{2}(I^{\otimes 4} - H{\otimes}H{\otimes}I{\otimes}I)|\psi\rangle \geq \sin^2(\frac{\pi}{8})$$

Global Energy Bound

Step 2: $|\psi\rangle$ is pseudo-stabilizer at $\Omega(n)$ local terms of $\frac{1}{m}\sum_{i=1}^{m_x} \frac{1}{2}(I - \mathrm{H}^{\otimes k})$!

T-count gives a lower bound on $|G|$, we give an upper bound which depends on the sizes of "locally-commuting" subgroups of G.

Lemma. There are subgroups of G, $\{G_i\}$, which (1) locally-commute at h_i, (2) are the largest subgroups of G with this property, and (3) satisfy

$$|G| \leq \prod_i |G_i|$$

Combined with the lower-bound, $|G| \geq 2^{(1-c)n}$, (3) implies that $\Omega(n)$ of the subgroups have size $|G_i| = 2^k$.

Future Directions

- Examine other Hamiltonian implications of QPCP:

- Improved hardness results for constant-gap LH: BQP-hardness, MA-hardness, …