Local Hamiltonians with No Low-Energy Stabilizer States

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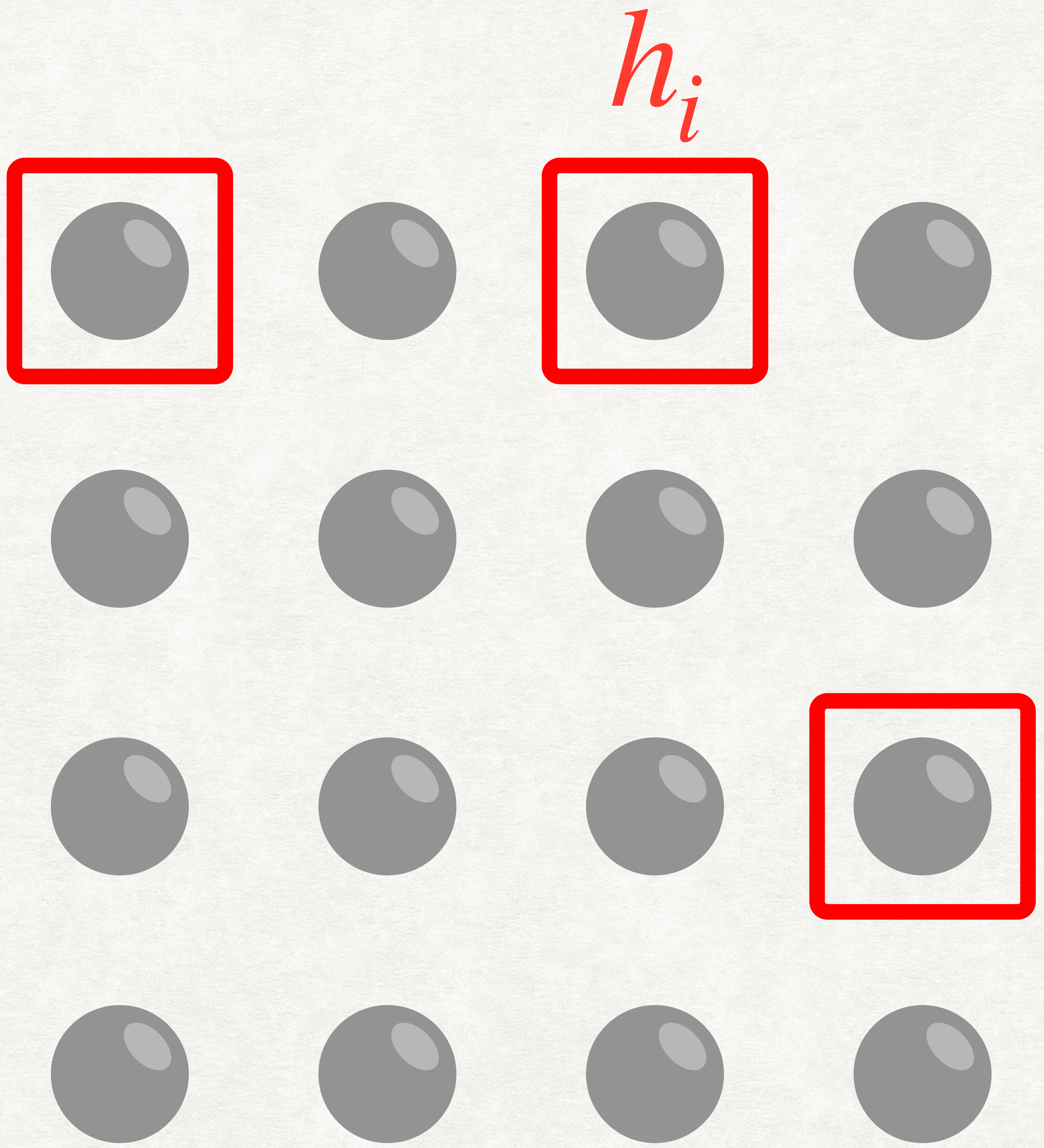
Local Hamiltonians

k-local interaction term: h_i PSD with $\|h_i\| \leq 1$



Local Hamiltonians

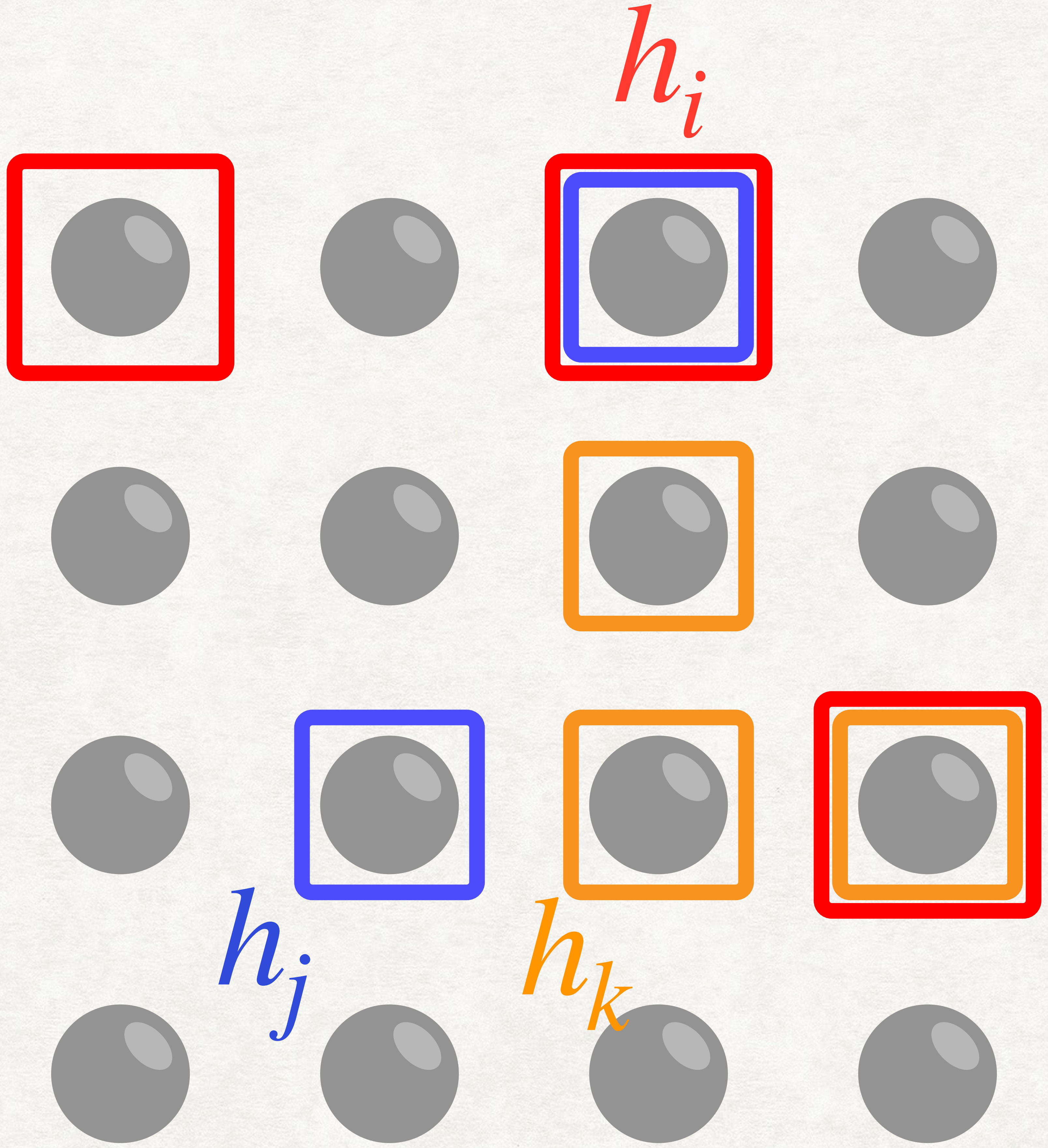
k-local interaction term: h_i PSD with $\|h_i\| \leq 1$

k-local Hamiltonian: $m = \text{poly}(n)$ k-local terms

$$H = \frac{1}{m} \sum_{i=1}^{m} h_i \otimes I_{2^{n-k}}$$

Ground-state energy: $E_{gs} = \min_{|\psi\rangle} \ \langle \psi | H | \psi \rangle$

Can we approximate E_{gs} to within some error $\delta(n)$ (in BQP)?



Local Hamiltonian Problem

Local Hamiltonian problem (LH): given H, $\epsilon > 0$, $\delta(n)$, decide between

$$(1)\ E_{gs} \leq \epsilon \quad \text{or} \quad (2)\ E_{gs} > \epsilon + \delta(n)\,.$$

For $\delta(n) = \dfrac{1}{\text{poly}(n)}$, LH is QMA-complete [KSV02].

⇒ **Can't approximate E_{gs} with error $\delta(n) = \dfrac{1}{\text{poly}(n)}$ unless BQP=QMA.**

qPCP Conjecture

Can we approximate E_{gs} with constant precision?

Quantum PCP Conjecture (qPCP): LH is QMA-hard with $\delta(n) = \Omega(1)$.

By classical PCP Theorem LH-$\Omega(1)$ is at least NP-hard.

What does a "qPCP Hamiltonian" look like?

If $C \subsetneq$ QMA = QPCP is conjectured $\Rightarrow$ cannot estimate $E_{gs} \pm \epsilon$ in C.

$\Rightarrow$ No ground state should have an energy estimation algorithm in C.

What do *low-energy states* look like?

If $C \subsetneq$ QMA = QPCP is conjectured $\Rightarrow$ cannot estimate $E_{gs} \pm \epsilon$ in C.

$\Rightarrow$ No *low-energy state* should have an energy estimation algorithm in C.

[Low-energy = small constant energy above E_{gs}]

If qPCP Conjecture is true, $\exists H$ whose low-energy states cannot admit energy approximation algorithms.

Some ways to estimate energy

1. Trivial (i.e. Low-depth circuit) states — NP via light cone argument
2. "Sampleable states" — MA via dequantizing QSVT [GL22]
3. Stabilizer states — NP via stabilizer generators
4. Matrix product states — NP via tensor contraction
5. …

Some ways to estimate energy

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A “qPCP” Hamiltonian can’t have any of these in its low-energy space.

Low-energy space implications

Hamiltonians that should exist if qPCP is true…

1. No low-energy trivial states — NLTS Theorem [ABN22]

2. No low-energy "sampleable states" — NLSS Conjecture [GL22]

3. No low-energy stabilizer states — NLCS Theorem (our work)

Note: Stabilizer states can be efficiently sampleable, so NLSS $\Rightarrow$ NLCS.

No Low-Energy Stabilizer States (NLCS)

Stabilizer state ⇔ prepared by a Clifford circuit (H, CNOT, S)

H **satisfies the** ϵ**-NLCS property if every stabilizer state has energy** $\langle\psi|H|\psi\rangle \geq \epsilon$**.***

Intuition: Ground-state energy can't be approximated using stabilizer states.

Theorem. There exists an explicit $\sin^2(\pi/8)$-NLCS Hamiltonian.

* if $E_{gs} = 0$

Simple NLCS Hamiltonian

Starting point: $H_0 \equiv \frac{1}{n} \sum |1\rangle\langle 1|_i$

Ground state: $|0^n\rangle$ with 0 energy

Main idea: rotate the ground-space into a basis which is highly *non-stabilizer*.

We consider the Y version of the T gate: $D \equiv e^{i\frac{\pi}{8}Y}$

Simple NLCS Hamiltonian

Rotated: $\tilde{H}_0 \equiv D^{\otimes n} H_0 D^{\dagger \otimes n} = \frac{1}{n} \sum D |1\rangle\langle 1|_i D^\dagger$

New ground state: $D^{\otimes n} |0^n\rangle$ with 0 energy

Theorem. $\tilde{H}_0$ is $\sin^2(\pi/8)$-NLCS.

Simple NLCS Hamiltonian

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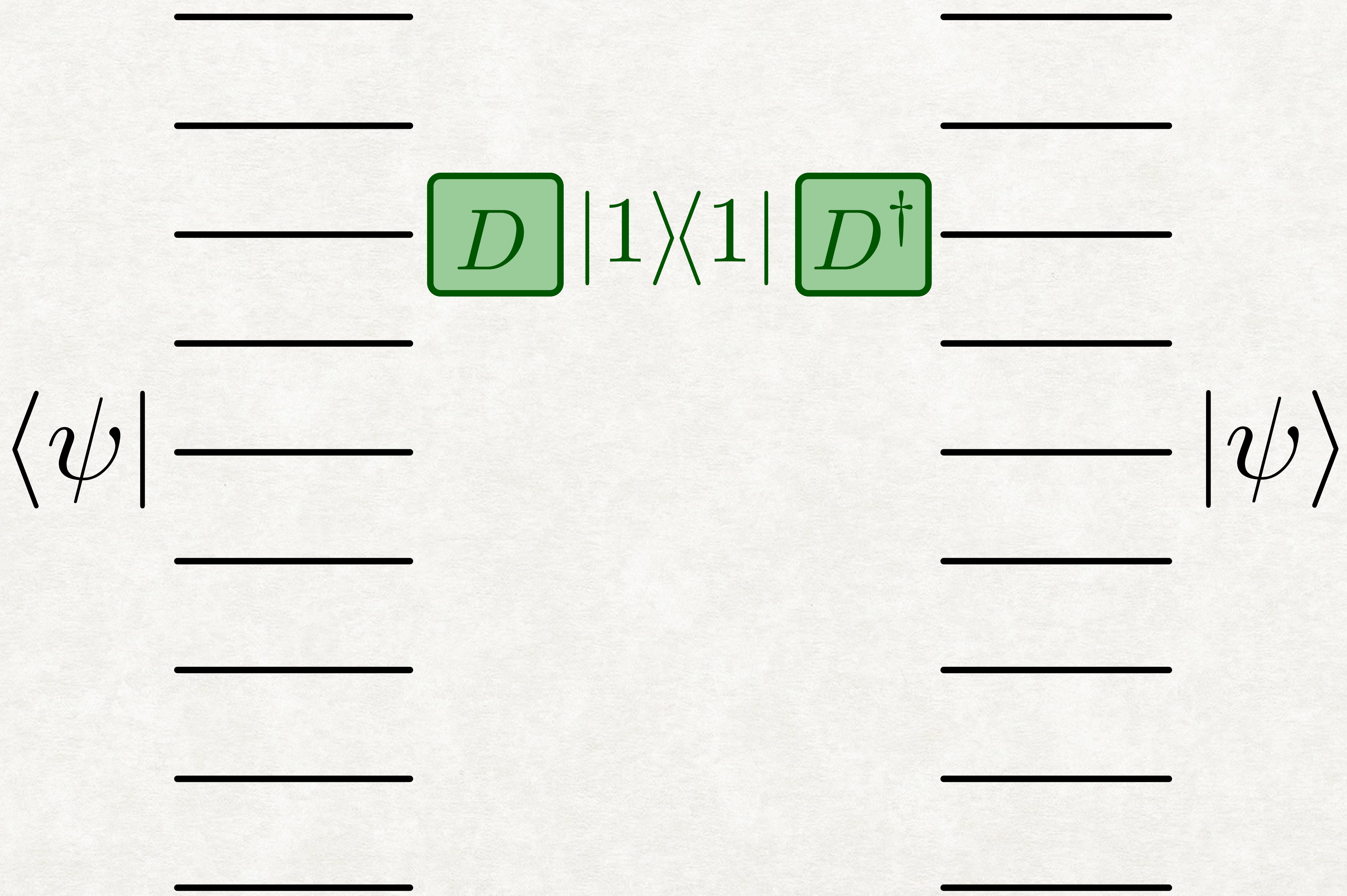
Proof. Consider a single term:



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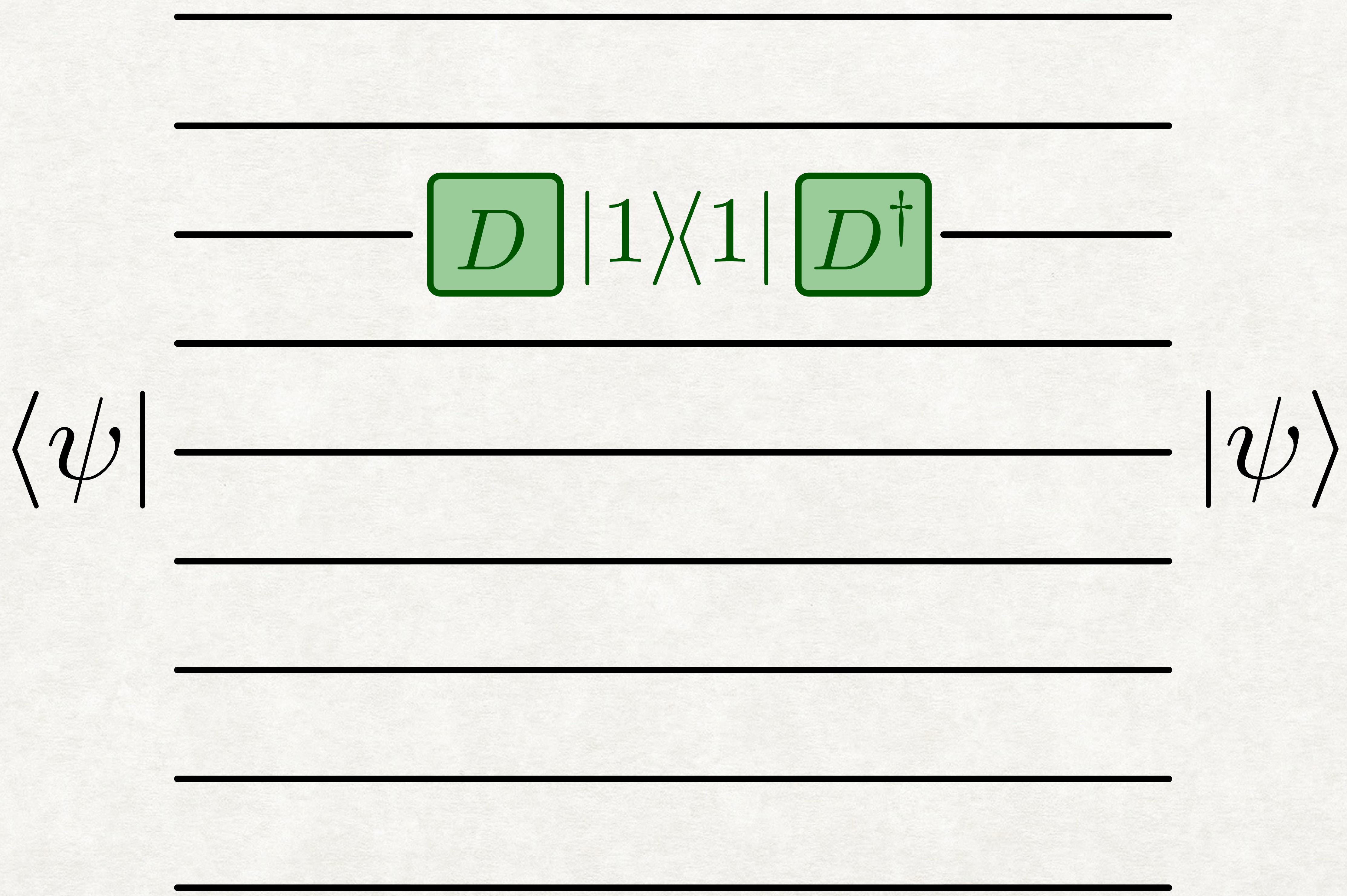

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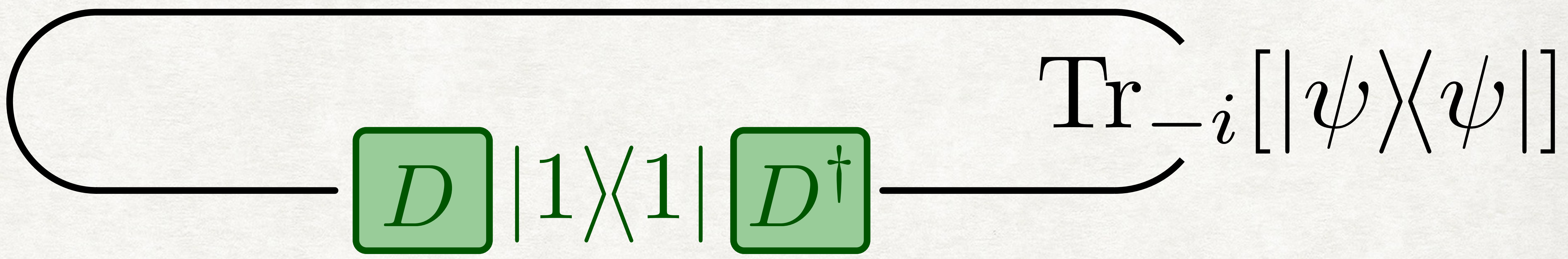

Fact: If $|\psi\rangle$ is an n-qubit stabilizer state then $\psi_i \equiv \mathrm{Tr}_{-i}[\,|\psi\rangle\langle\psi|\,]$ is either $\frac{1}{2}I$ or a stabilizer state $|\eta\rangle\langle\eta|$.

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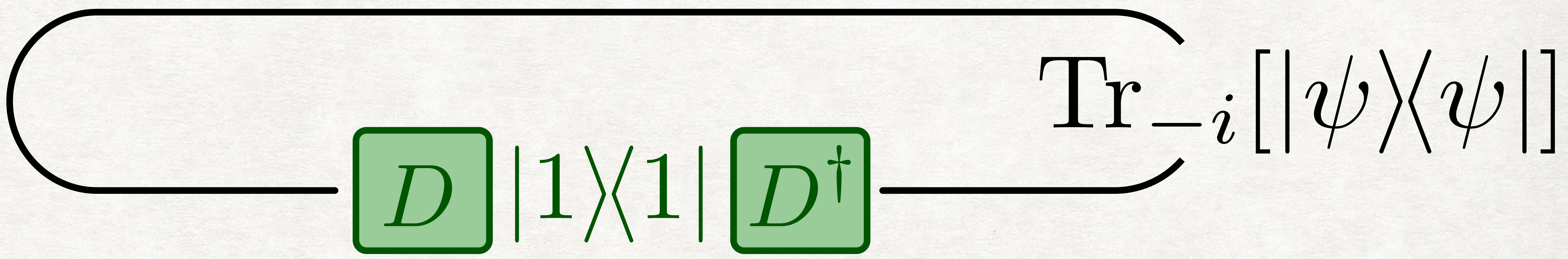

So:

$$\langle\psi|D|1\rangle\langle 1|_i D^\dagger|\psi\rangle = \frac{1}{2}$$

or $= |\langle\eta|D|1\rangle|^2$

Simple NLCS Hamiltonian

Theorem. $\tilde{H}_0$ is $\sin^2(\pi/8)$-NLCS.

Proof.

Direct computation: $|\langle \eta | D | 1 \rangle|^2 \geq \sin^2(\pi/8)$ for all single-qubit stabilizer states.

Since all local terms are lower-bounded by $\sin^2(\pi/8)$ we're done.

Combinations of types

Can we get a *simultaneous* NLTS and NLCS Hamiltonian?

Well… $H_{NLTS} \otimes I + I \otimes H_{NLCS}$, but this isn't a very interesting system.

H_{NLTS} is a CSS Hamiltonian, i.e. it's ground-space corresponds to a stabilizer code.

Can we get NLTS+NLCS directly from rotating by $D = e^{i\frac{\pi}{8}Y}$?

CSS Hamiltonians

Theorem. For CSS Hamiltonians, $\tilde{H} \equiv D^{\otimes n} H D^{\dagger \otimes n}$ is $\sin^2(\pi/8)$-NLCS.

Corollary. $\tilde{H}_{NLTS}$ satisfies both NLTS and NLCS.

Why? H_{NLTS} is a CSS Hamiltonian, and rotating by a constant-depth circuit preserves NLTS.

CSS Hamiltonians

Theorem. For CSS Hamiltonians, $\tilde{H} \equiv D^{\otimes n} H D^{\dagger \otimes n}$ is $\sin^2(\pi/8)$-NLCS.

Proof components: Let $|\psi\rangle$ be an n-qubit stabilizer state.

(1) k-local states of $|\psi\rangle$ are mixtures of k-qubit stabilizer states $\Rightarrow$ sufficient to bound local energy terms for k-qubit stabilizer states.

(2) The local terms of $\tilde{H}$ look like $\dfrac{I - \mathrm{H}^{\otimes k}}{2}$ or $\dfrac{I - (\text{-XHX})^{\otimes k}}{2}$ (H = Hadamard) $\Rightarrow$ local lower bound follows from an upper bound on $|\langle \eta | \mathrm{H}^{\otimes k} | \eta \rangle|$.

(3) Main technical lemma: For all k-qubit stabilizer states $\Rightarrow |\langle \eta | \mathrm{H}^{\otimes k} | \eta \rangle| \leq \dfrac{1}{\sqrt{2}}$.

* if k is odd

Future Work

- We showed NLCS+NLTS Hamiltonians exist. What if we relax the stabilizer requirement to Clifford + a single T gate? $\log(n)$ T gates? (No Low-Energy "Almost Clifford" States)

"Almost Clifford" states

Take $\tilde{H}_0$ to be the rotated zero Hamiltonian.

Conjecture. Suppose $|\psi\rangle$ can be prepared with $\leq \alpha\, T$ gates. Then

$$\langle\psi|\tilde{H}_0|\psi\rangle \geq \left(1 - \frac{\alpha}{n}\right)\sin^2(\pi/8)\,.$$

"Almost Clifford" states

Take $\tilde{H}_0$ to be the rotated zero Hamiltonian.

NLACS Theorem (unpublished). Suppose $|\psi\rangle$ can be prepared with $\leq \alpha\, T$ gates. Then

$$\langle\psi|\,\tilde{H}_0\,|\psi\rangle \geq \left(1 - \frac{\alpha}{n}\right) \sin^2(\pi/8)\,.$$

Corollary. Low-energy states of $\tilde{H}_0$ require $n - o(n)\ T$ gates.

$\Rightarrow E_{gs}$ energy can't be approximated using stabilizer + $\log(n)\ T$ gate states

Note: $\tilde{H}_0$ still has an NP witness

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- NLSS Conjecture? Suggestion: $CH_0C^\dagger$ for a low-depth circuit C.
- Witness state "lower-bounds", i.e. Hamiltonians with no low-energy:
  1. "stabilizer then low-depth" states (rotated stabilizer Hamiltonians have NP witnesses)
  2. states with classical descriptions that can be used to compute k-local reduced density matrices (NLLS Conjecture)
- Complexity lower-bounds for constant-gap LH problem (BQP-hardness, MA-hardness, etc.)

Thanks!

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Simple NLACS Hamiltonian

Rotated zero Hamiltonian: $\tilde{H}_0 \equiv \frac{1}{n}\sum D|1\rangle\langle 1|_i D^\dagger \sim \frac{1}{n}\sum \frac{I - \mathsf{H}_i}{2}$

Energy of a single term: $\frac{1}{2}\left(1 - \langle\psi|\mathsf{H}_i|\psi\rangle\right)$

If $S \in \mathrm{Stab}(|\psi\rangle)$ acts non-trivially on i, then $\langle\psi|\mathsf{H}_i|\psi\rangle \leq \frac{1}{\sqrt{2}}$

If $|\psi\rangle$ can be prepared with α T gates, then $n - \alpha$ qubits are acted on non-trivially by a stabilizer.