Hamiltonians whose low-energy states require $\Omega(n)$ T gates

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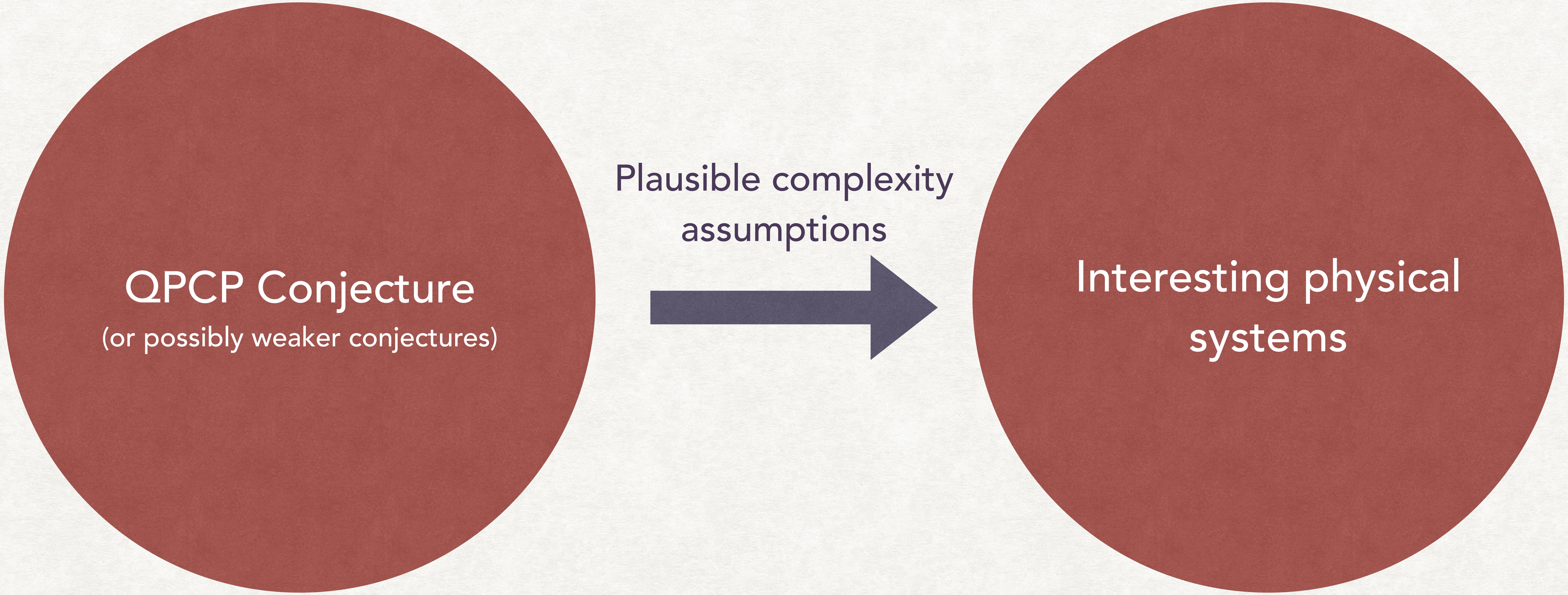
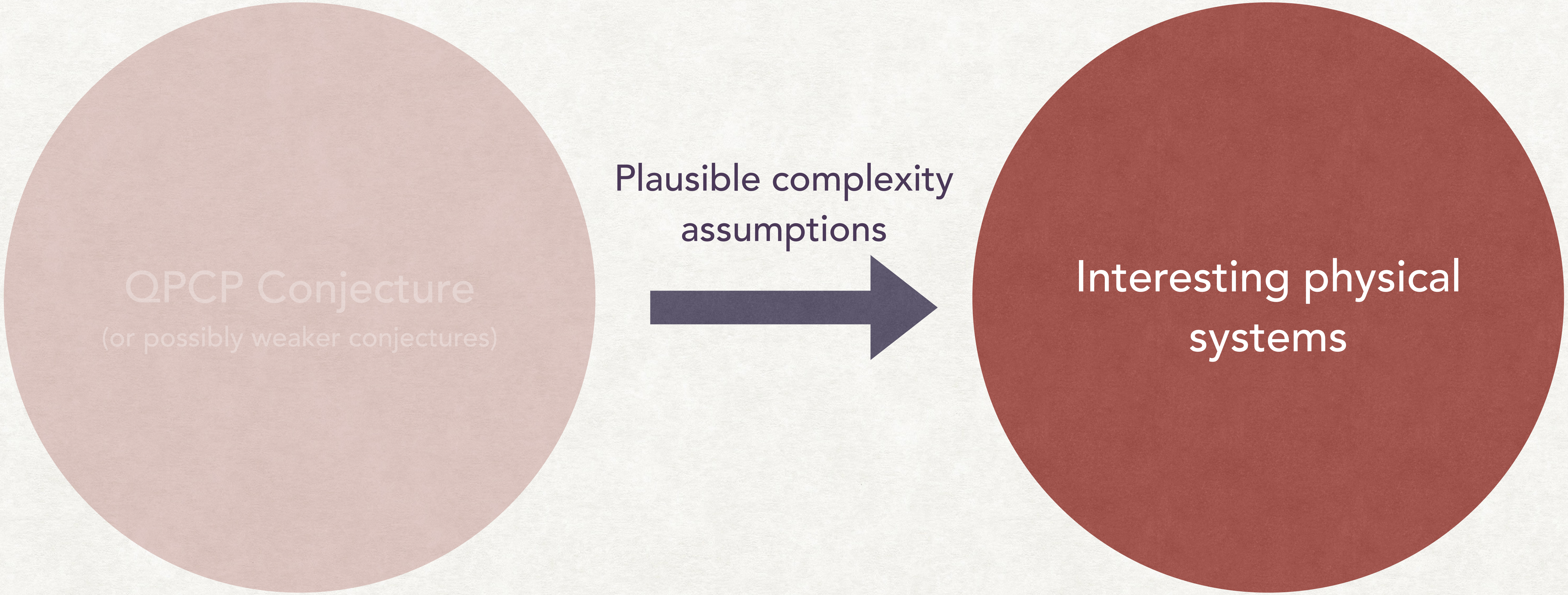

QPCP Conjecture
(or possibly weaker conjectures)
Plausible complexity assumptions
Interesting physical systems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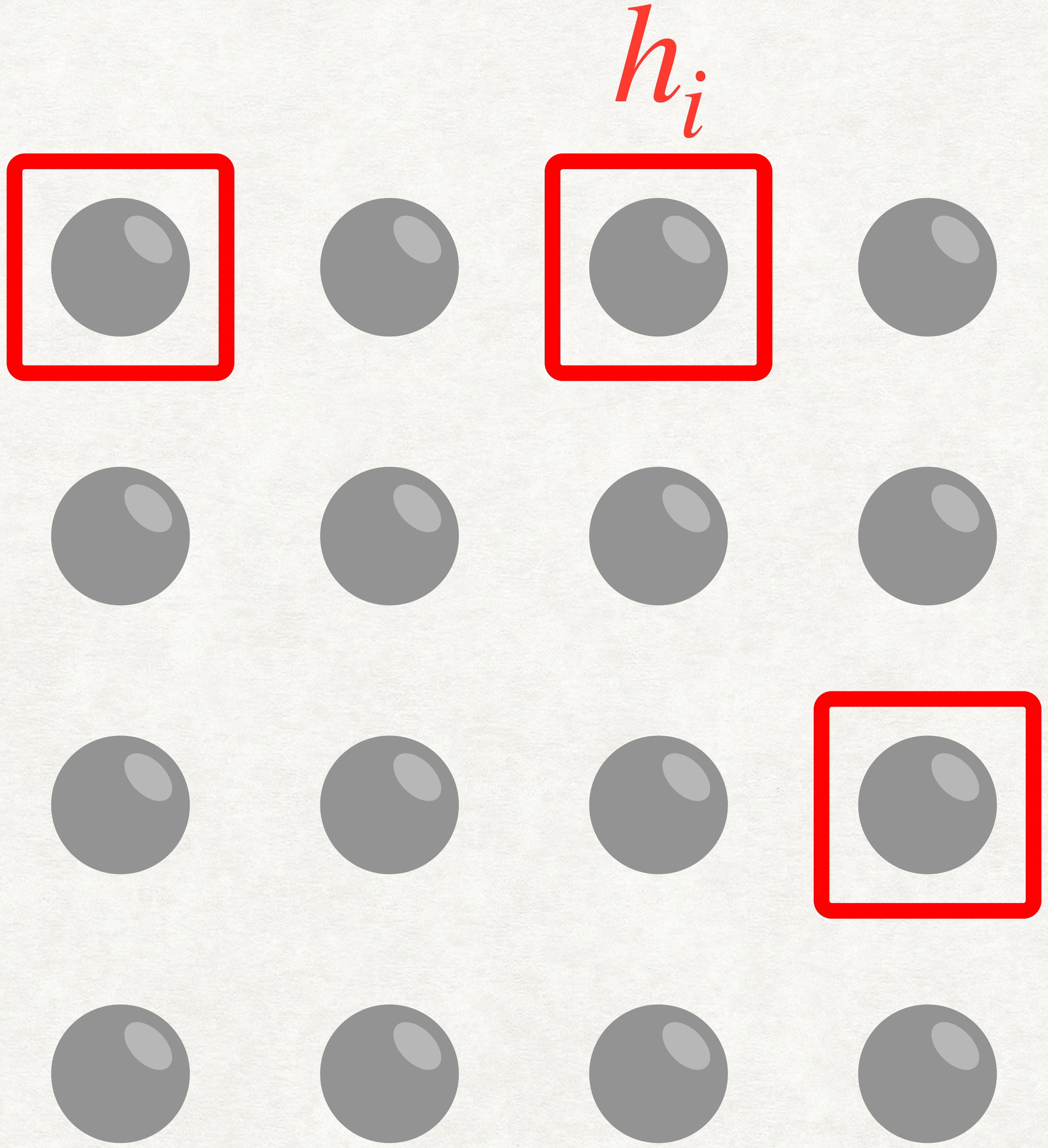
Plausible complexity assumptions
Interesting physical systems

Outline

- Quantum complexity basics
- Implications of QPCP
- Simple NLACS Hamiltonian
- CSS Hamiltonians and joint NLTS/NLACS
- Future directions

Local Hamiltonians

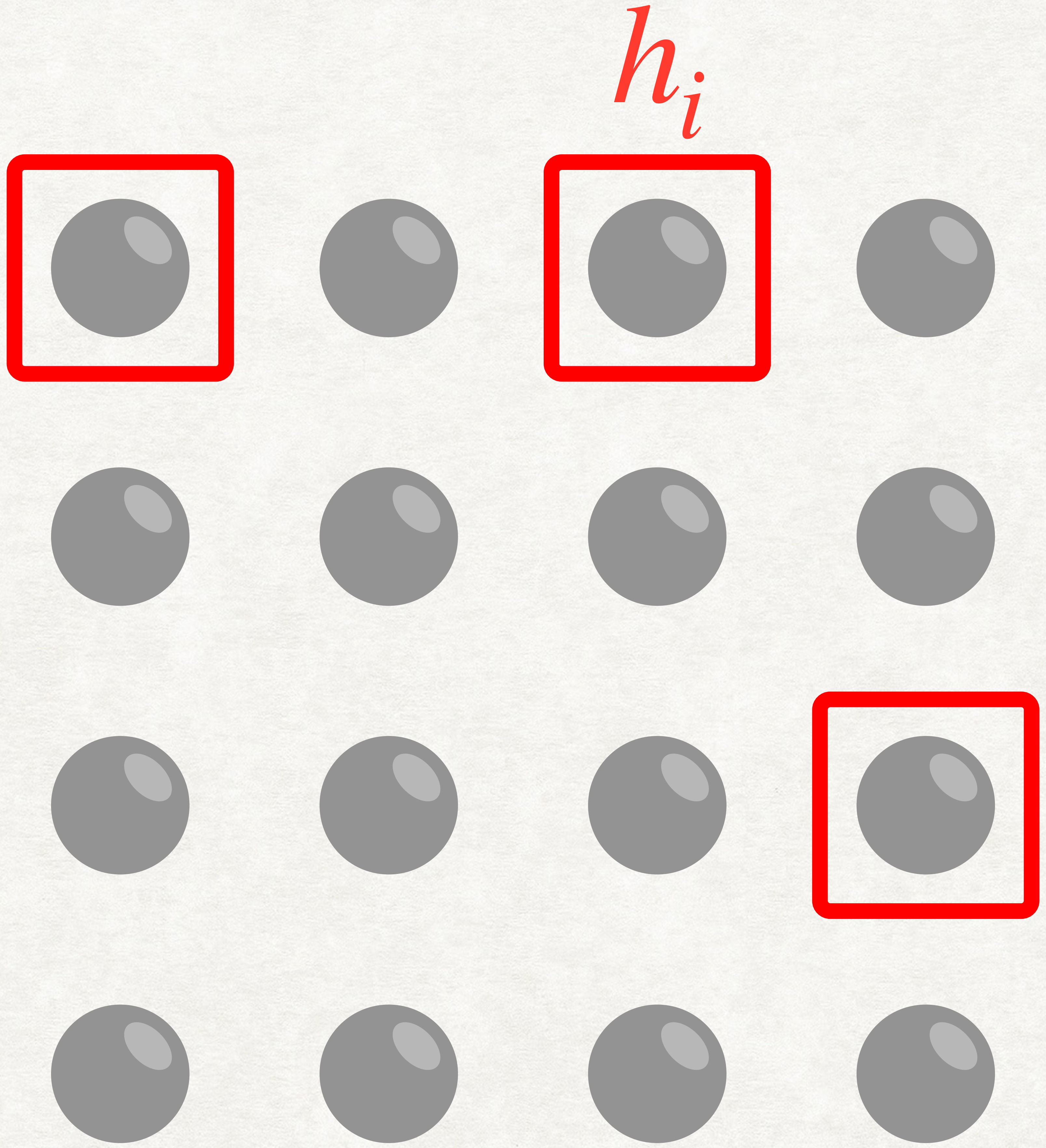
k-local interaction term: h_i PSD with $\|h_i\| \leq 1$



Local Hamiltonians

k-local interaction term: h_i PSD with $\|h_i\| \leq 1$

$h_i \otimes I_{2^{n-k}}$



Local Hamiltonians

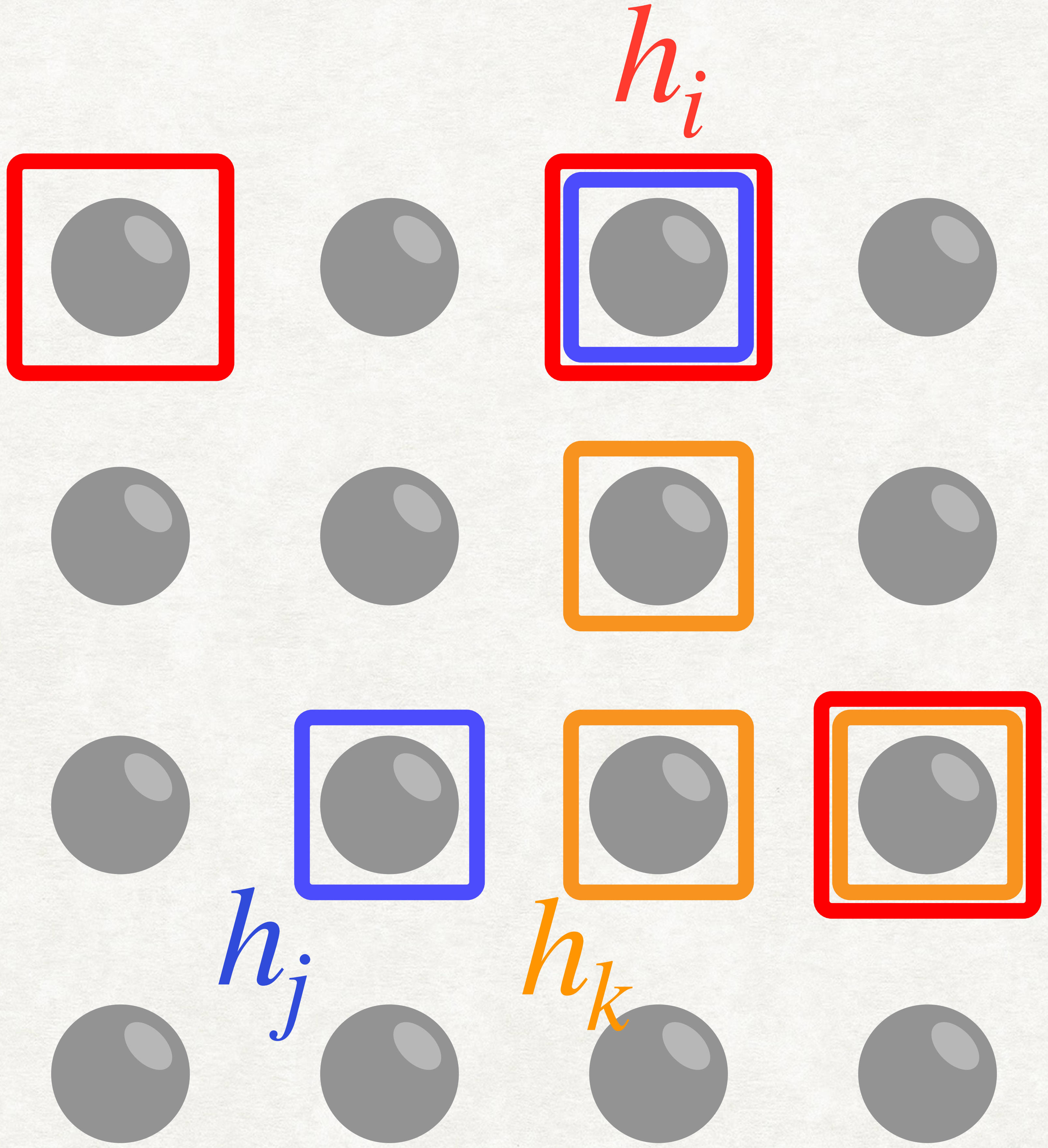
k-local interaction term: h_i PSD with $\|h_i\| \leq 1$

k-local Hamiltonian: $m = \text{poly}(n)$ k-local terms

$$H = \frac{1}{m} \sum_{i=1}^{m} h_i \otimes I_{2^{n-k}}$$

Ground-state energy: $E_{gs} \equiv \min_{|\psi\rangle} \langle \psi | H | \psi \rangle$

Can we approximate E_{gs} to within some error $\epsilon(n)$ (in BQP)?



Local Hamiltonian Problem

k-Local Hamiltonian problem (LH-ϵ): given H, a, $\epsilon(n) > 0$, decide between

$$(1)\ E_{gs} \leq a \quad \text{or} \quad (2)\ E_{gs} > a + \epsilon(n)\,.$$

where $E_{gs} \equiv \min_{|\psi\rangle} \langle \psi | H | \psi \rangle$.

Computing $E_{gs} \pm \epsilon/2 \Rightarrow$ solution to LH-ϵ!

Complexity classes

A decision problem is in QMA (Quantum Merlin Arthur) if there is an efficient *quantum* algorithm which can verify solutions to the problem using a *quantum* witness state.

$$V\big(|x\rangle \otimes |\psi\rangle\big) = \begin{cases} 1 & \text{w.h.p if the answer is yes} \\ 0 & \text{w.h.p if the answer is no} \end{cases}$$

$|\psi\rangle,\ poly(|x|)$ qubit state

Complexity classes

A decision problem is in MA (Merlin Arthur) if there is an efficient *probabilistic* algorithm which can verify solutions to the problem using a *classical* witness state.

$$V(x, y) = \begin{cases} 1 & \text{w.h.p if the answer is yes} \\ 0 & \text{w.h.p if the answer is no} \end{cases}$$

$y, \; poly(|x|)$ length bit string

Complexity classes

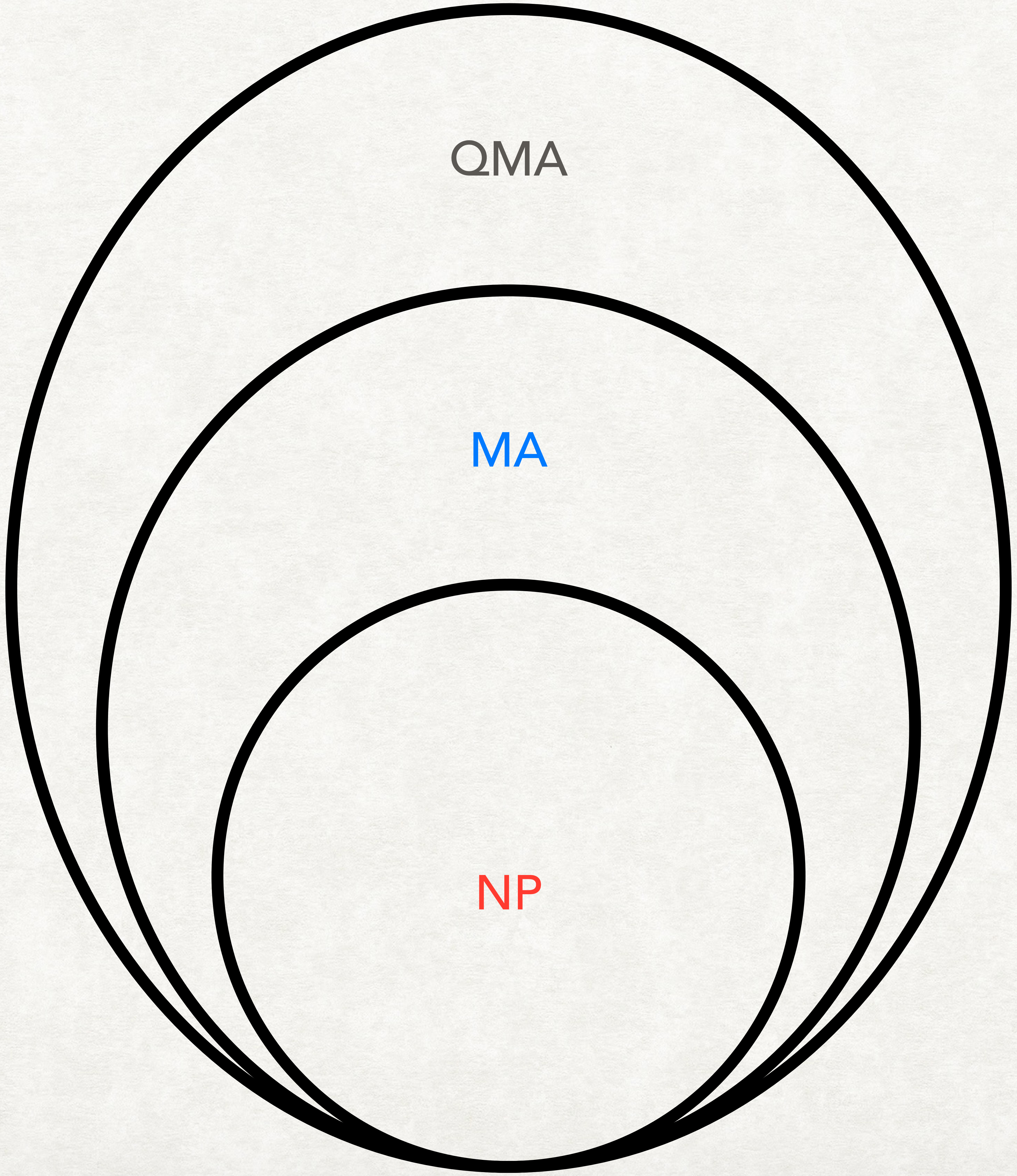
A decision problem is in NP if there is an efficient *deterministic* algorithm which can verify solutions to the problem using a *classical* witness state.

$$V(x,y) = \begin{cases} 1 & \text{if the answer is yes} \\ 0 & \text{if the answer is no} \end{cases}$$

y, $poly(|x|)$ length bit string

Complexity classes

Widely believed that NP and MA are not equal to QMA!



Local Hamiltonian Problem

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where $E_{gs} \equiv \min_{|\psi\rangle} \langle\psi|H|\psi\rangle$.

(Quantum Cook–Levin) For $\epsilon(n) = \dfrac{1}{\text{poly}(n)}$, LH-$\epsilon$ is QMA-complete [KSV02].

Classically: MAX-k-SAT is NP-complete for $\epsilon(n) = \dfrac{1}{\text{poly}(n)}$.

PCP Theorem

k-Local Hamiltonian problem (LH-ϵ): given H, a, $\epsilon(n) > 0$, decide between

$$(1)\ E_{gs} \le a \quad \text{or} \quad (2)\ E_{gs} > a + \epsilon(n)\,.$$

where $E_{gs} \equiv \min_{|\psi\rangle} \langle \psi | H | \psi \rangle$.

(Quantum Cook–Levin) For $\epsilon(n) = \dfrac{1}{\mathrm{poly}(n)}$, LH-$\epsilon$ is QMA-complete [KSV02].

Classically: MAX-k-SAT is NP-hard for $\epsilon(n) = \Omega(1)$.

QPCP

Quantum PCP Conjecture (QPCP): LH-ϵ is QMA-hard for $\epsilon(n) = \Omega(1)$.

By classical PCP Theorem LH-$\Omega(1)$ is at least NP-hard.

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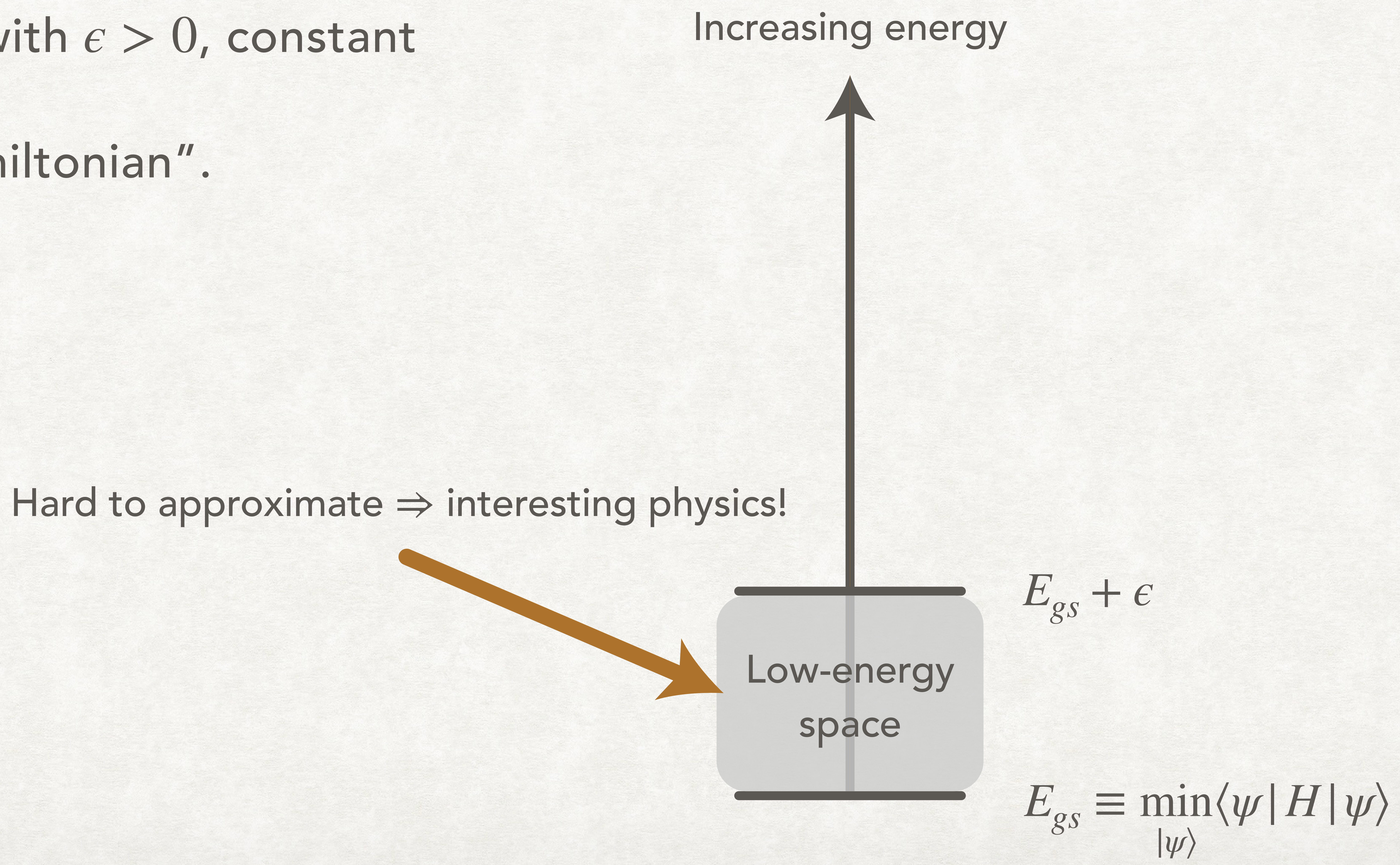
Implications on Hamiltonian Complexity

Assume QPCP is true with $\epsilon > 0$, constant

Let H be a "QPCP Hamiltonian".



Energy estimation

Let $C =$ NP or MA. An n-qubit state, $|\psi\rangle$, admits "energy estimation in C" if:

(1) $|\psi\rangle$ has an *efficient* classical description, $desc(|\psi\rangle) \in \{0,1\}^{poly(n)}$.

(2) There is a C-verifier*, W, for which $\left| W\big(H, desc(|\psi\rangle)\big) - \langle\psi|H|\psi\rangle \right| \leq \epsilon = O(1)$

Only special types of quantum states admit classical energy estimation

Sampleable states

$|\psi\rangle$ is a sampleable state if:

(1) $|\psi\rangle$ has an *efficient* classical description, $desc(|\psi\rangle) \in \{0,1\}^{poly(n)}$.

(2) There is a classical algorithm using $desc(|\psi\rangle)$ to compute amplitudes, $\langle x|\psi\rangle$.

(3) There is a classical algorithm using $desc(|\psi\rangle)$ to sample from $p(x) = |\langle x|\psi\rangle|^2$.

Energy estimation in MA via dequantization of QSVT [GL22]

Stabilizer (or Clifford) states

$\mathscr{P}_1 \equiv \{I, X, Y, Z\}$, $\mathscr{P}_n \equiv \mathscr{P}_1^{\otimes n}$, e.g., $X \otimes Y \otimes I \otimes Z \in \mathscr{P}_4$

Stabilizer group: $\mathrm{Stab}(|\psi\rangle) = \{P \in \mathscr{P}_n \mid P|\psi\rangle = |\psi\rangle\}$

Stabilizer state: $|\mathrm{Stab}(|\psi\rangle)| = 2^n$ ($\Leftrightarrow$ prepared by a Clifford circuit)

By the Gottesman–Knill Theorem, stabilizer states are efficiently sampleable.

Energy estimation in NP via stabilizer generators

Almost-Clifford states

$\mathscr{P}_1 \equiv \{I, X, Y, Z\}$, $\mathscr{P}_n \equiv \mathscr{P}_1^{\otimes n}$, e.g., $X \otimes Y \otimes I \otimes Z \in \mathscr{P}_4$

Stabilizer group: $\mathrm{Stab}(|\psi\rangle) = \{P \in \mathscr{P}_n \mid P|\psi\rangle = |\psi\rangle\}$

Almost-Clifford state: $|\mathrm{Stab}(|\psi\rangle)| \geq 2^{n-\log n}$ ($\Leftrightarrow$ prepared by Clifford + $O(\log n)$ T gates)

By extensions of Gottesman-Knill, almost-Clifford states are efficiently sampleable

Energy estimation in NP via linear combination of stabilizer states

Low-energy space implications

Let H be a "QPCP Hamiltonian" and C a complexity class. Assuming $C \subsetneq$ QMA $\Rightarrow$ cannot estimate $E_{gs} \pm \epsilon$ in C

$\Rightarrow$ No *low-energy state* should have an energy estimation algorithm in C.

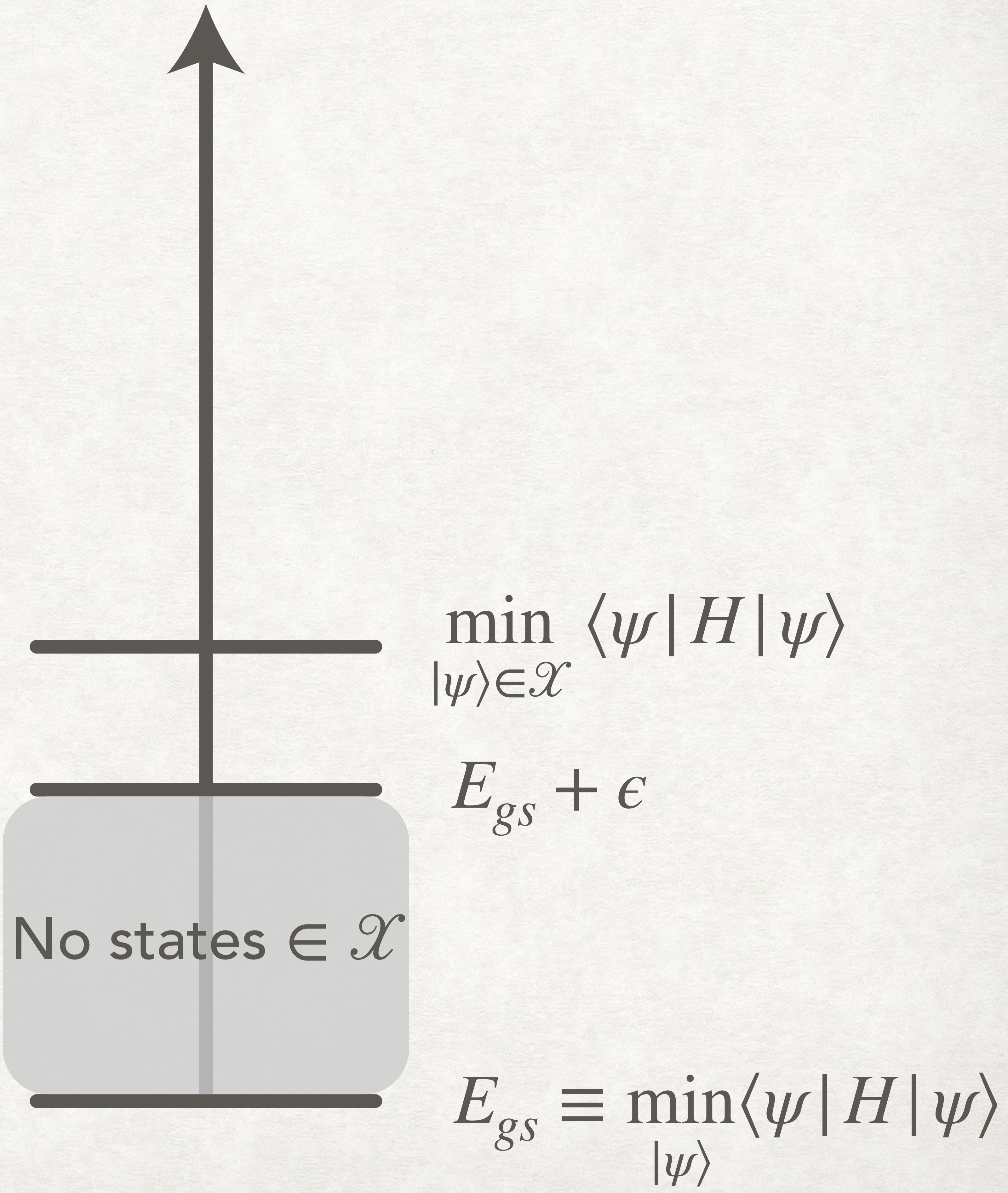
Why? Approximating energies of arbitrary low-energy states in $C \Rightarrow$ LH-$\Omega(1) \in C$

Low-energy space implications

Let $\mathscr{X}$ be a class of states with energy estimation in C.

QPCP + $C \subsetneq$ QMA $\implies$ there is an H and constant $\epsilon > 0$ s.t. $\min_{|\psi\rangle \in \mathscr{X}} \langle\psi|H|\psi\rangle \geq E_{gs} + \epsilon$.

Such an H is said to satisfy the No Low-energy $\mathscr{X}$ States (NL$\mathscr{X}$S) property.



Low-energy space implications

Hamiltonians that should exist if QPCP is true…

1. No low-energy trivial states — NLTS Theorem [Anshu, Breuckmann, Nirkhe 22]

2. _____________ "sampleable states" — NLSS Conjecture [Gharibian, Le Gall 22]

3. _____________ stabilizer states — NLCS Theorem [**C**, Coudron, Nelson, Nezhadi 23a]

4. _____________ almost-Clifford states — NLACS Theorem [**C**CNN23b]

5. _____________ locally-approximately states — NLLS Conjecture [**C**CNN23a, WFG23]

…

No Low-Energy Almost-Clifford States (NLACS)

H **satisfies the** ϵ**-NLACS property if every almost-Clifford state has energy**

$$\langle\psi|H|\psi\rangle \geq \epsilon.\text{*}$$

Fact. Every NLSS Hamiltonian is an NLACS Hamiltonian.

* when $E_{gs} = 0$

Can we construct such Hamiltonians *independently* of QPCP?

Main Results

Theorem. There exists an explicit local Hamiltonian satisfying $\alpha \sin^2(\pi/8)$-NLACS for every $\alpha \in (0,1)$.

Theorem. There exists an explicit local Hamiltonian simultaneously satisfying NLACS and NLTS.

(In fact, low-energy states require $n - o(1)$ T gates)

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Simple NLACS Hamiltonian

Starting point: $H_+ \equiv \frac{1}{n}\sum |-\rangle\langle -|_i$

Ground state: $|+\rangle^{\otimes n}$ with 0 energy

Main idea: rotate the ground-space into a basis which is highly *non-stabilizer*.

We consider the Y version of the T gate: $D \equiv e^{i\frac{\pi}{8}Y}$

Simple NLACS Hamiltonian

Rotated: $H_D \equiv D^{\otimes n} H_+ D^{\dagger \otimes n} = \frac{1}{n} \sum D | - \rangle \langle - |_i D^\dagger$

New ground state: $D^{\otimes n} | + \rangle^{\otimes n}$ with 0 energy

Ground state has no stabilizers: $Stab(D^{\otimes n} | + \rangle^{\otimes n}) = \{I\}$

Simple NLACS Hamiltonian

Rotated: $H_D \equiv D^{\otimes n} H_+ D^{\dagger \otimes n} = \frac{1}{n} \sum \frac{I - \mathrm{H}_i}{2}$, H = Hadamard

New ground state: $D^{\otimes n} | + \rangle^{\otimes n}$ with 0 energy

Ground state has no stabilizers: $Stab(D^{\otimes n} | + \rangle^{\otimes n}) = \{I\}$

NLACS Theorem

Theorem [CCNN23b]. If $|\psi\rangle$ can be prepared by Clifford + $\leq \alpha$ T gates, then

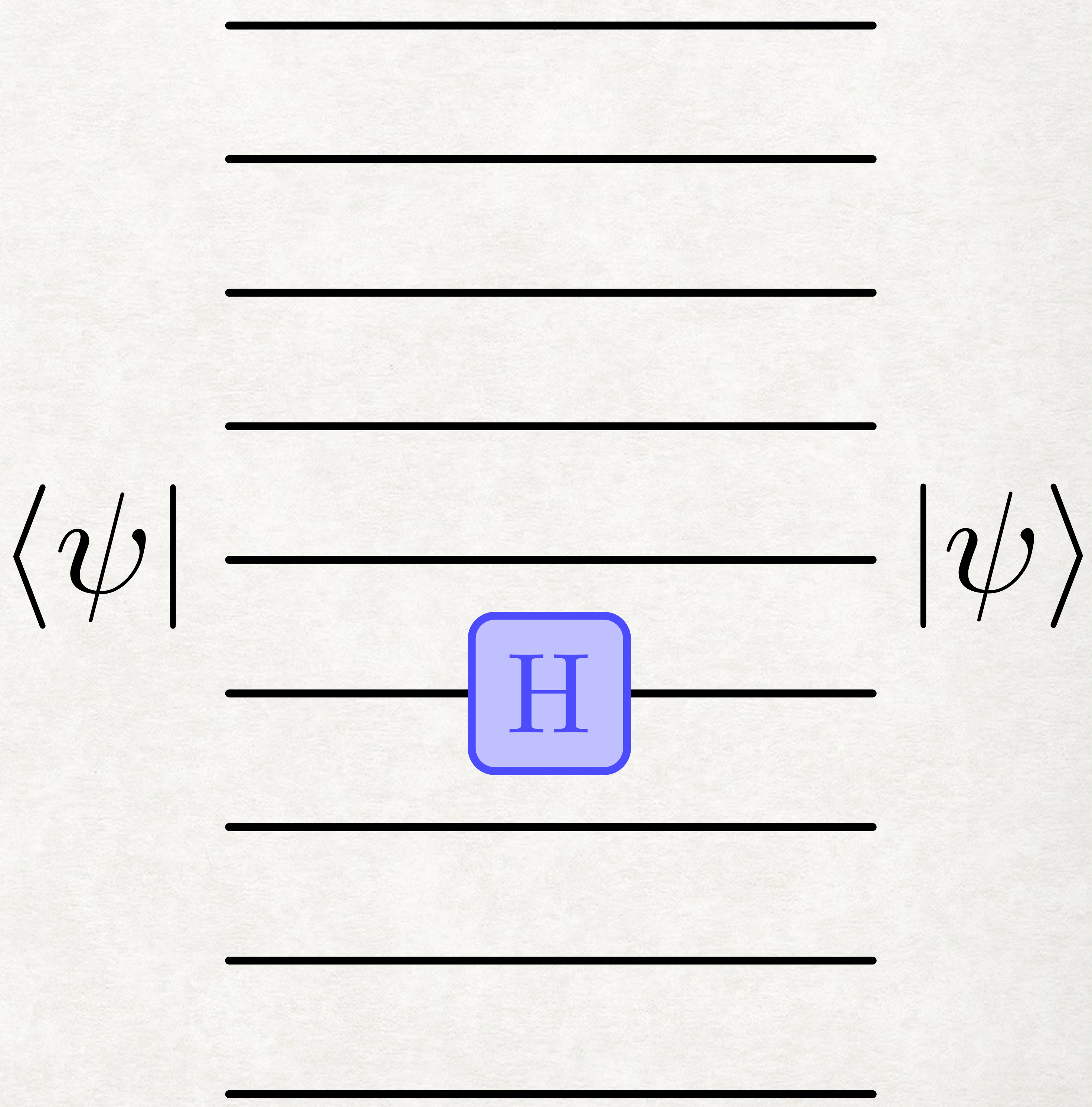
$$\langle\psi|H_D|\psi\rangle \geq \left(1 - \frac{\alpha}{n}\right)\sin^2\left(\frac{\pi}{8}\right)$$

Intuition: need $\alpha \sim n$ T gates to have arbitrary low energy.

Corollary. For every $c \in (0,1)$, H_D is $c\sin^2(\pi/8)$-NLACS.

Local bound — single term

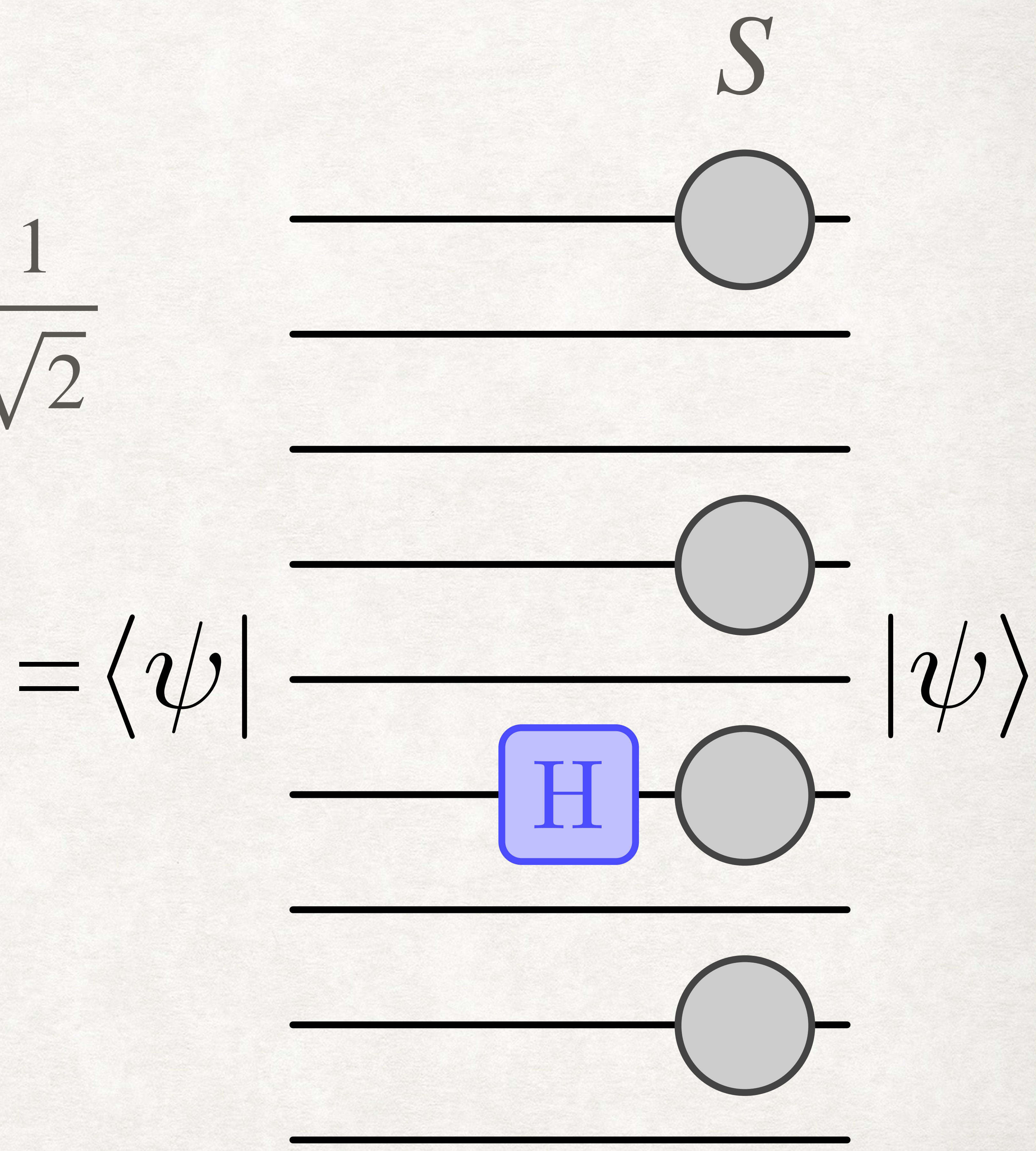
$$E_i = \frac{1}{2}\Big(1 - \langle\psi|\,\mathsf{H}_i\,|\psi\rangle\Big)$$



Local bound — single term

$$E_i = \frac{1}{2}\Big(1 - \langle\psi|\,\mathsf{H}_i\,|\psi\rangle\Big)$$

Lemma. $S \in \mathrm{Stab}(\,|\psi\rangle)$ acts non-trivially on $i \Rightarrow \langle\psi|\,\mathsf{H}_i\,|\psi\rangle \le \frac{1}{\sqrt{2}}$



Fact 1. $Stab(\mathsf{H}_i\,|\psi\rangle) = \mathsf{H}_i\; Stab(\,|\psi\rangle)\;\mathsf{H}_i$

Fact 2. If $|\psi\rangle$ and $|\varphi\rangle$ have anti-commuting stabilizers, then

$$|\langle\psi|\varphi\rangle| \le \frac{1}{\sqrt{2}}$$

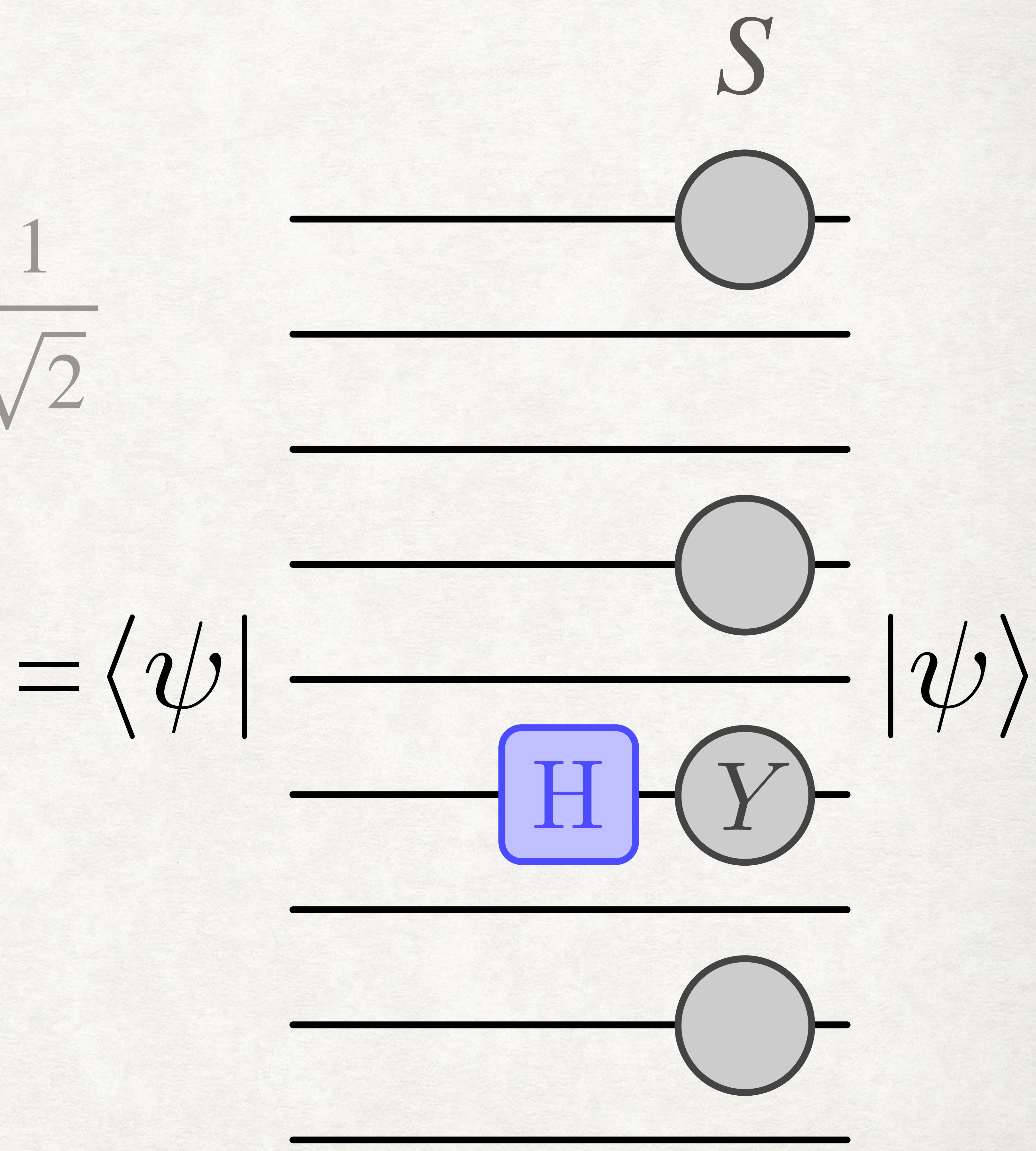
Local bound — single term

$$E_i = \frac{1}{2}\Big(1 - \langle\psi|\,\mathsf{H}_i\,|\psi\rangle\Big)$$

Lemma. $S \in \mathrm{Stab}(|\psi\rangle)$ acts non-trivially on $i \Rightarrow \langle\psi|\,\mathsf{H}_i\,|\psi\rangle \leq \frac{1}{\sqrt{2}}$

Proof.

If $S_i = Y$



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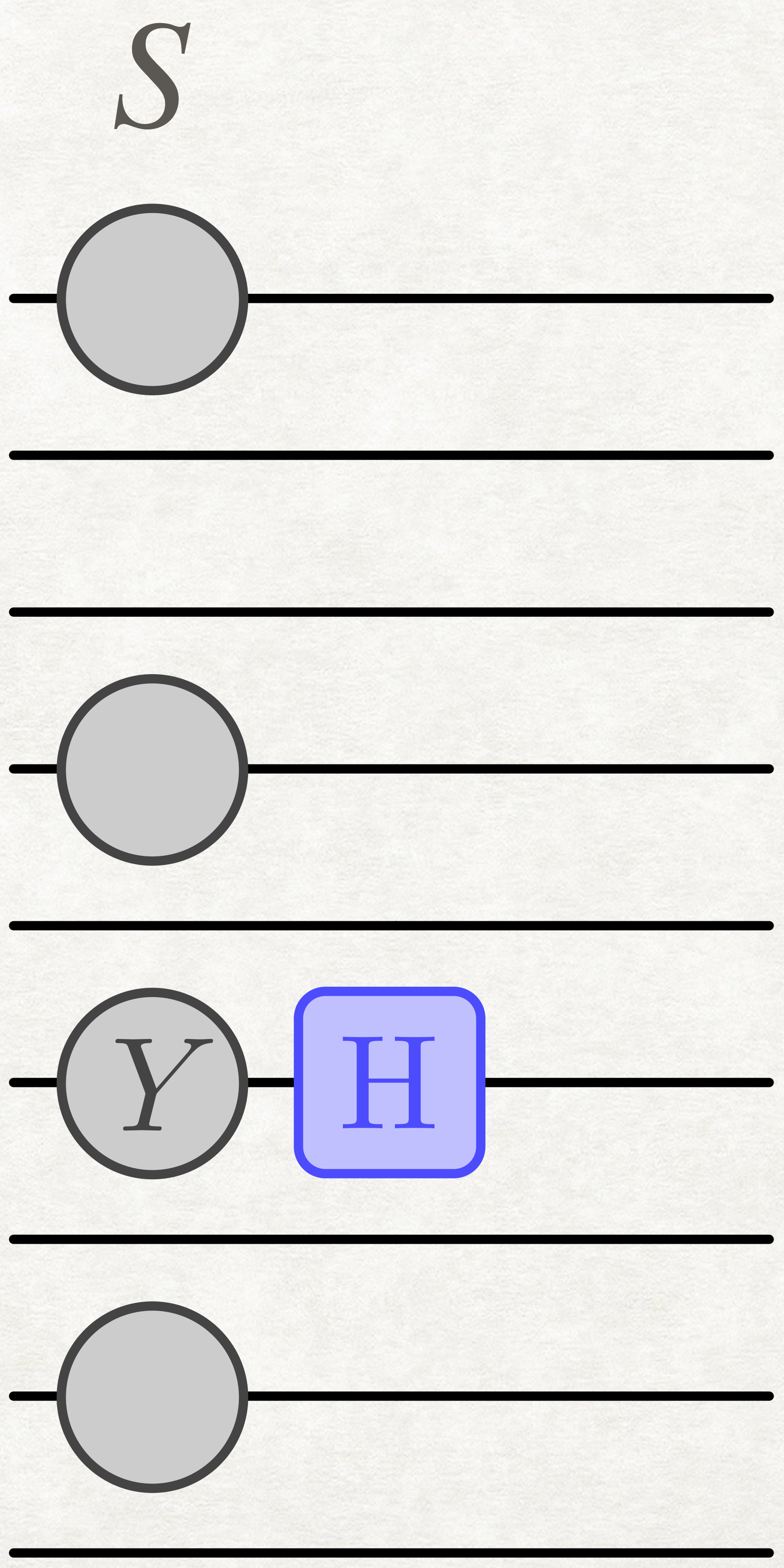
$= -\,\langle\psi|$  $|\psi\rangle$

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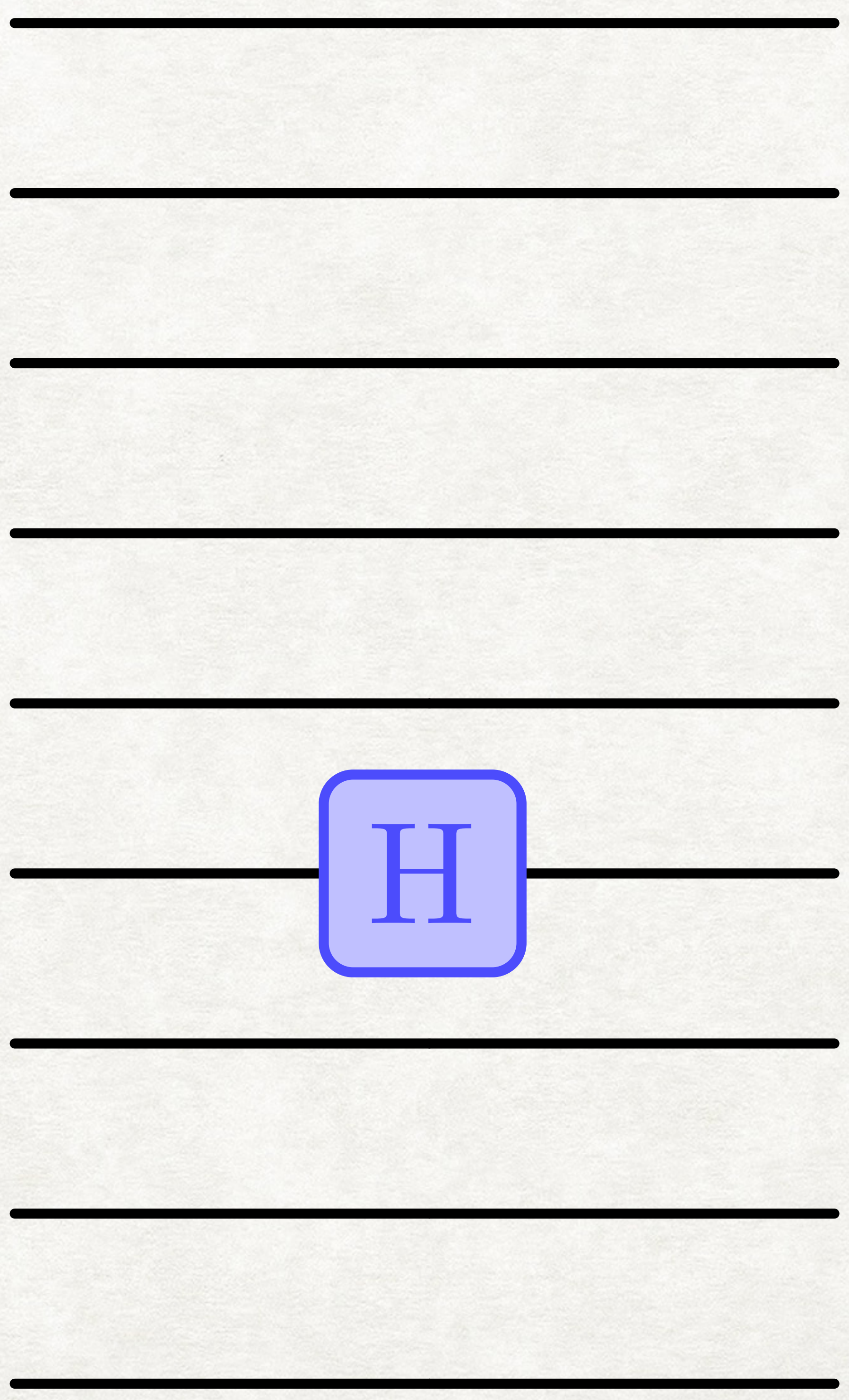
$= -\,\langle\psi|$  $|\psi\rangle$

$= 0$

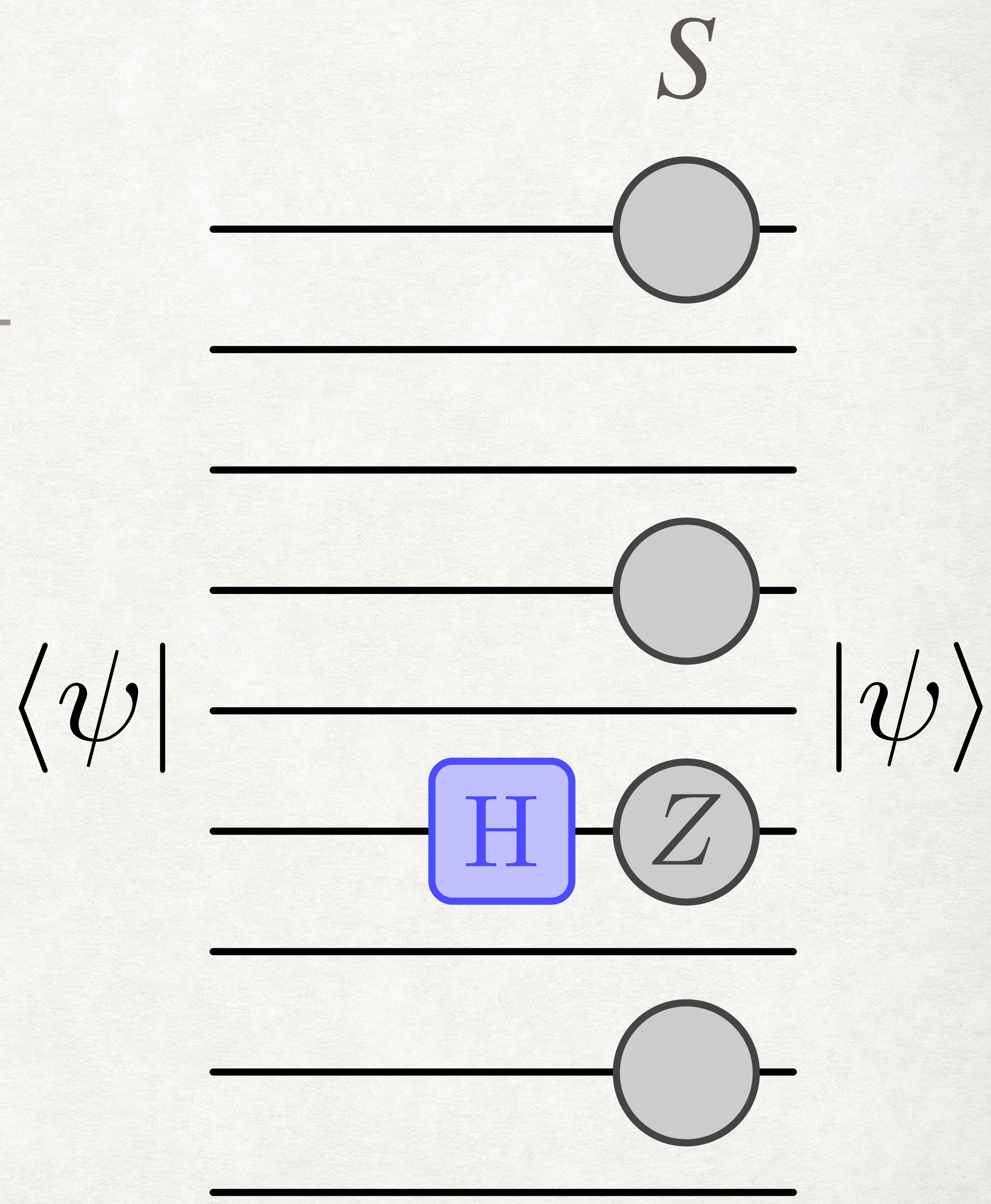
Local bound — single term

$$E_i = \frac{1}{2}\Big(1 - \langle\psi|\,\mathsf{H}_i\,|\psi\rangle\Big)$$

Lemma. $S \in \mathrm{Stab}(|\psi\rangle)$ acts non-trivially on $i \Rightarrow \langle\psi|\,\mathsf{H}_i\,|\psi\rangle \leq \frac{1}{\sqrt{2}}$

Proof.

If $S_i = Z$



Local bound — single term

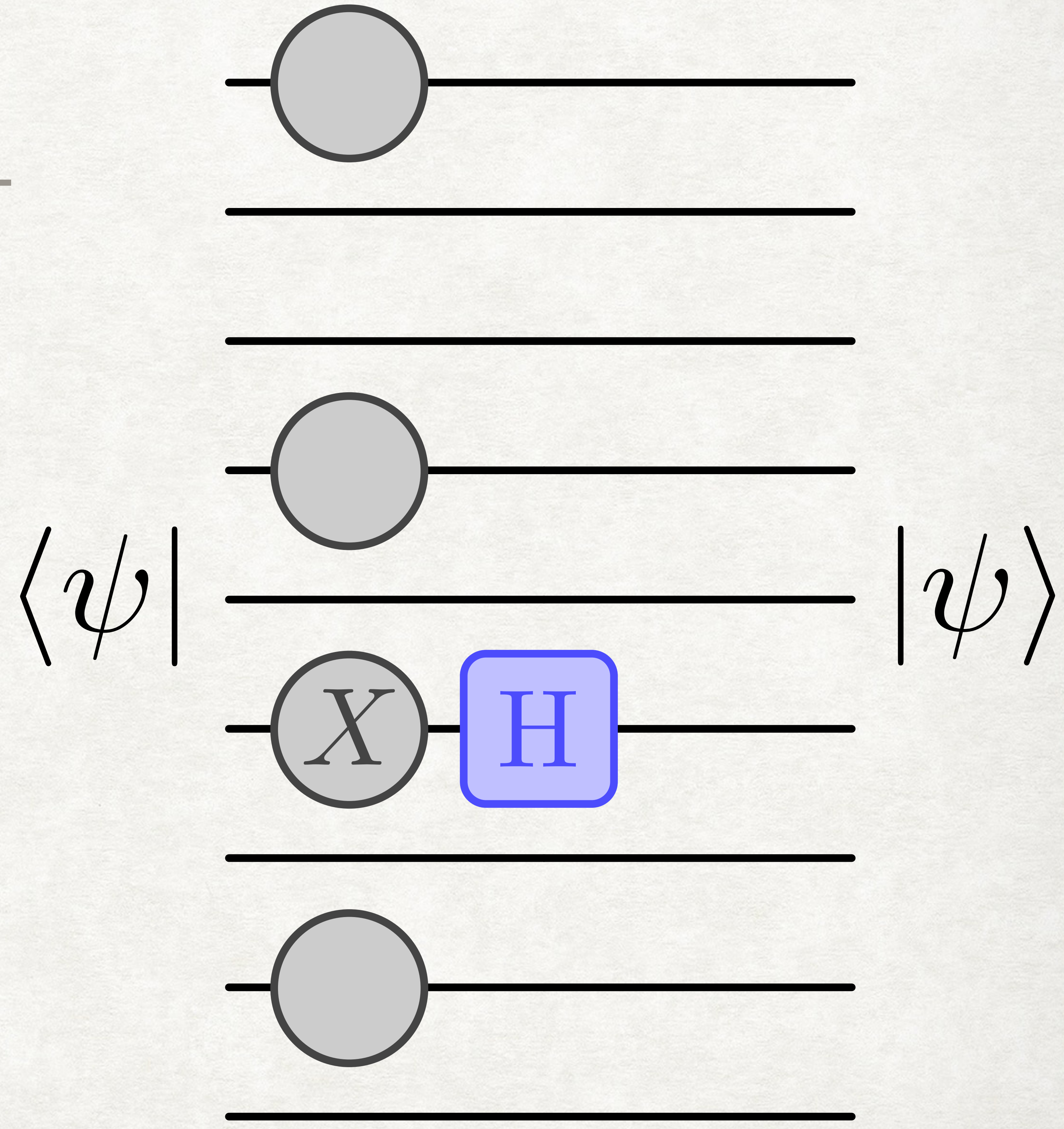
$$E_i = \frac{1}{2}\Big(1 - \langle\psi|\,\mathsf{H}_i\,|\psi\rangle\Big)$$

Lemma. $S \in \mathrm{Stab}(|\psi\rangle)$ acts non-trivially on $i \Rightarrow \langle\psi|\,\mathsf{H}_i\,|\psi\rangle \leq \frac{1}{\sqrt{2}}$

Proof.

If $S_i = Z$ then $|\psi\rangle$ and $\mathsf{H}_i|\psi\rangle$ have anti-commuting stabilizers.

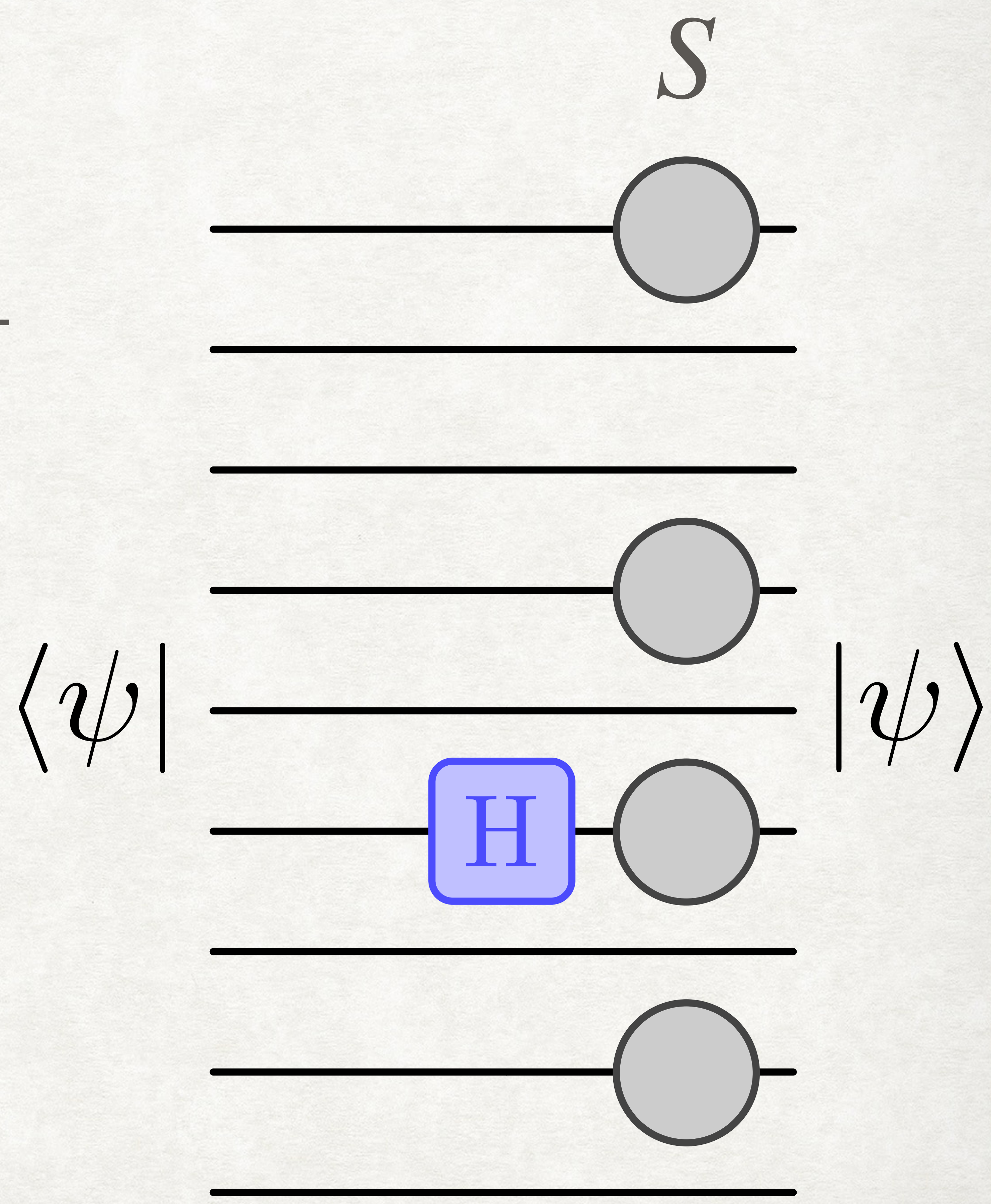
By Fact 2 the bound holds.



Local bound — single term

$$E_i = \frac{1}{2}\Big(1 - \langle\psi|\mathsf{H}_i|\psi\rangle\Big) \geq \frac{1}{2}\Big(1 - \frac{1}{\sqrt{2}}\Big) = \sin^2\Big(\frac{\pi}{8}\Big)$$

Lemma. $S \in$ Stab($|\psi\rangle$) acts non-trivially on $i \Rightarrow \langle\psi|\mathsf{H}_i|\psi\rangle \leq \frac{1}{\sqrt{2}}$



How many terms are acted on non-trivially?

Global bound — how many terms?

Lemma. $|\psi\rangle$, prepared by $\leq \alpha$ T gates $\Rightarrow \geq n - \alpha$ qubits are acted on non-trivially.

Proof idea:

1. $|Stab(|\psi\rangle)| \geq 2^{n-\alpha}$

2. If $Stab(|\psi\rangle)$ acted non-trivially on $< n - \alpha$ qubits $\Rightarrow |Stab(|\psi\rangle)| < 2^{n-\alpha}$. $\Rightarrow \Leftarrow$

Global bound — how many terms?

Lemma. $|\psi\rangle$, prepared by $\leq \alpha$ T gates $\Rightarrow \geq n - \alpha$ qubits are acted on non-trivially.

$$\Rightarrow \langle\psi|\, H_D \,|\psi\rangle \geq \left(\frac{n-\alpha}{n}\right)\sin^2\!\left(\frac{\pi}{8}\right)$$

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Joint NLTS/NLACS

1. Trivial (i.e. Low-depth circuit) states — NP via light cone argument
2. "Sampleable states" — MA via dequantizing QSVT [GL22]
3. Stabilizer states — NP via stabilizer generators
4. Almost-Clifford States — NP via linear-combination of stabilizer states
5. …

A "QPCP Hamiltonian" *simultaneously* can't have any of these in its low-energy space.

Joint NLTS/NLACS

1. No low-energy trivial states — NLTS Theorem [Anshu, Breuckmann, Nirkhe 22]
2. ______________ "sampleable states" — NLSS Conjecture [Gharibian, Le Gall 22]
3. ______________ stabilizer states — NLCS Theorem [C, Coudron, Nelson, Nezhadi 23a]
4. ______________ almost-Clifford states — NLACS Theorem [CCNN23b]
5. ______________ locally-approximately states — NLLS Conjecture [CCNN23a, WFG??]

Joint NLTS/NLACS

1. From CSS Hamiltonians — NLTS Theorem [Anshu, Breuckmann, Nirkhe 22]
2. ____________ "sampleable states" — NLSS Conjecture [Gharibian, Le Gall 22]
3. Rotated CSS Hamiltonians — NLCS Theorem [C, Coudron, Nelson, Nezhadi 23a]
4. Rotated CSS Hamiltonians — NLACS Theorem [CCNN23b]
5. ____________ locally-approximately states — NLLS Conjecture [CCNN23a, WFG??]

CSS Hamiltonians

$$H = \frac{1}{m}\sum_{i=1}^{m} \frac{I - S_i^{\otimes k}}{2}\Big|_{A_i},\ S_i \in \{X, Z\},$$ all terms commute.

$|\psi\rangle \in$ ground space $\Leftrightarrow \{S_i^{\otimes k}|_{A_i}\} \subseteq \mathrm{Stab}(|\psi\rangle)$

$m = \Theta(n) \Rightarrow$ ground-states are *highly* stabilized.

$k = O(1) \Rightarrow$ ground space is a *CSS QLDPC code*

CSS Hamiltonians

Theorem [ABN22]. There an explicit family of QLDPC CSS Hamiltonians, H, which has the NLTS property.

Can we rotate NLTS Hamiltonians so that they become NLACS?

CSS Hamiltonians

Theorem [ABN22]**.** There an explicit family of QLDPC CSS Hamiltonians, H, which has the NLTS property.

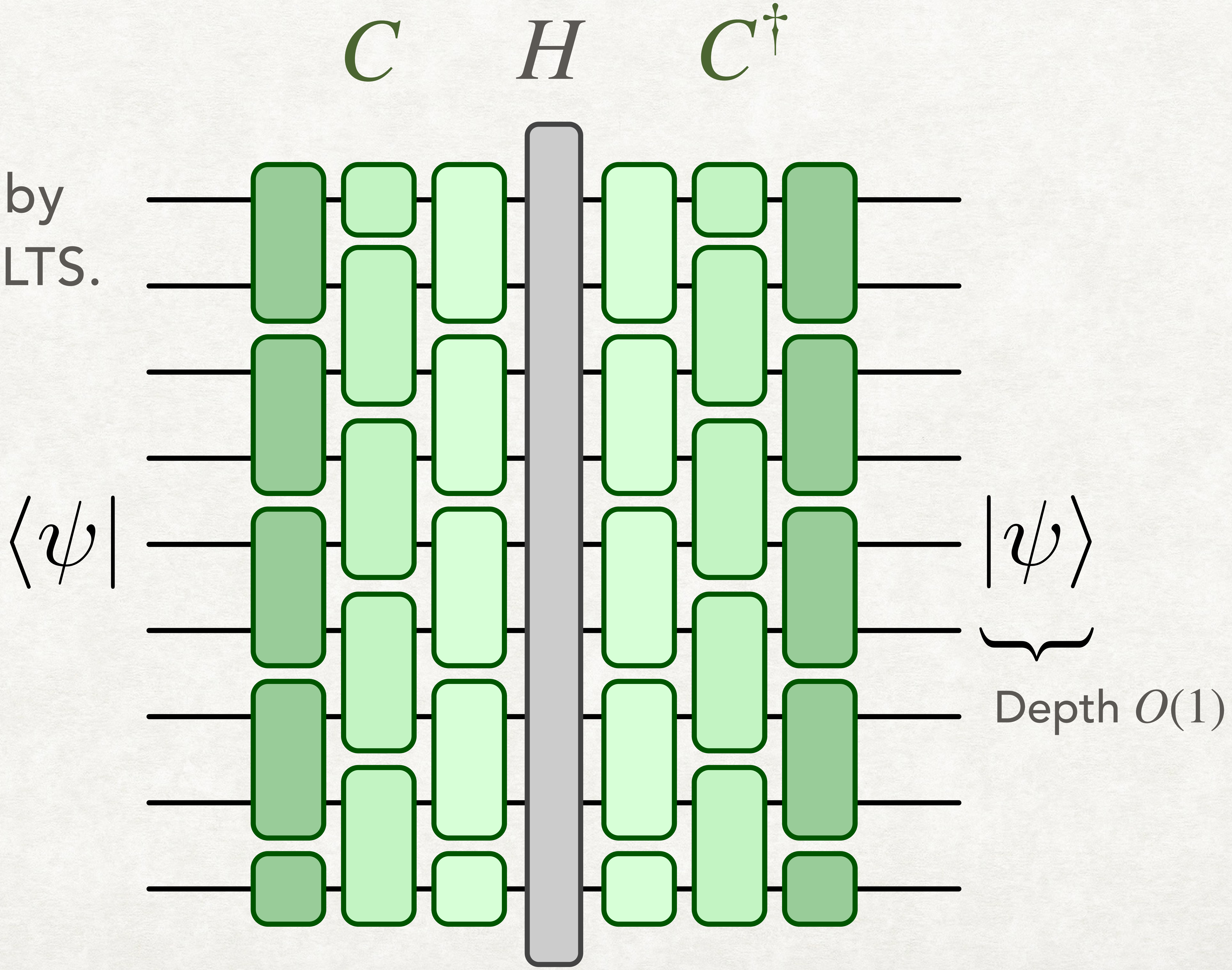
Theorem. If H is a QLDPC CSS Hamiltonian which satisfies NLTS, then $\tilde{H} \equiv D^{\otimes n} H D^{\dagger \otimes n}$ *simultaneously* satisfies NLTS and NLACS.

(In fact, low-energy states require $\Omega(n)$ T gates)

Rotating CSS ⇒ NLACS

Proof components.

Step 0. Rotating a local Hamiltonian by a constant-depth circuit preserves NLTS.



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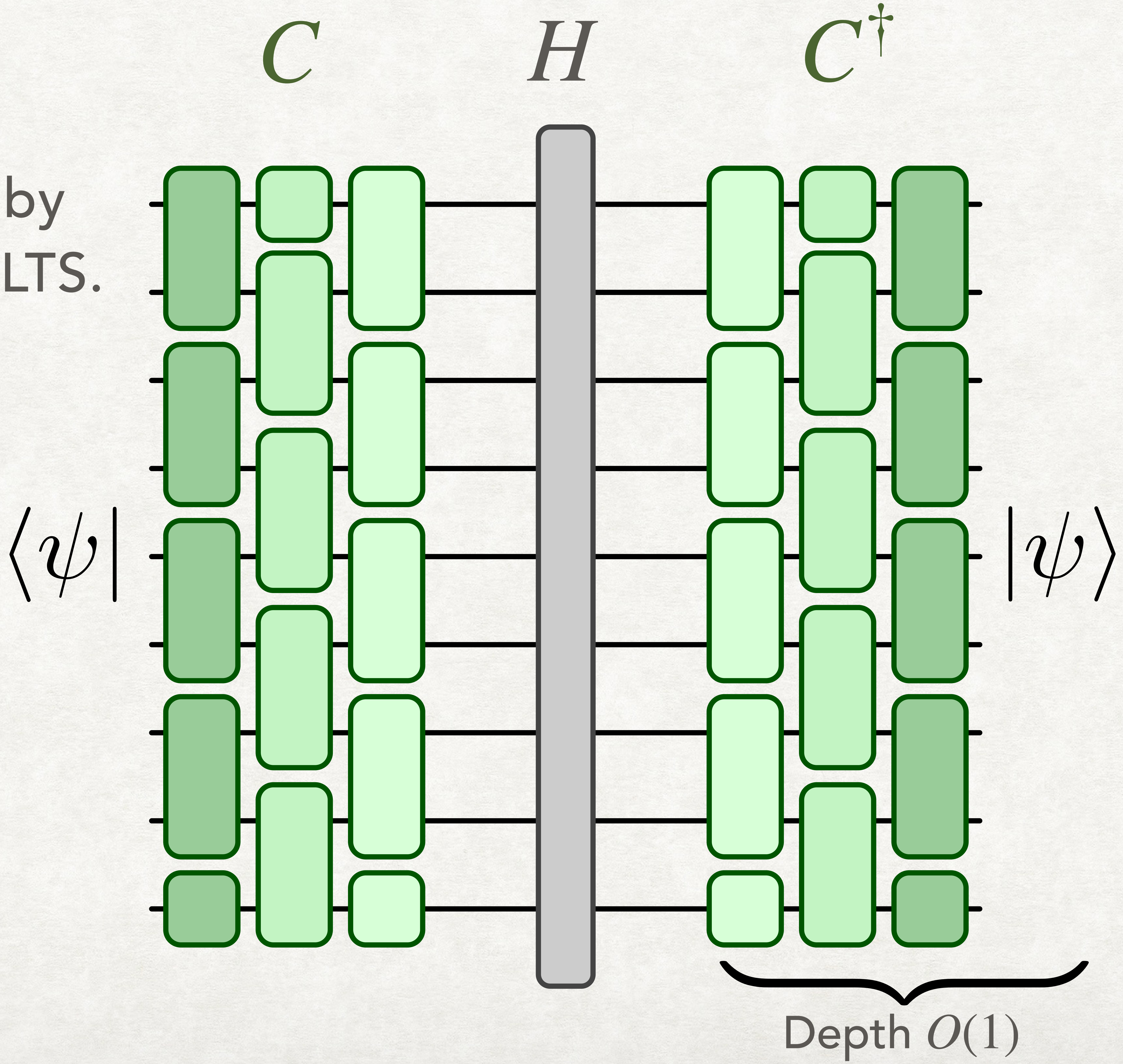
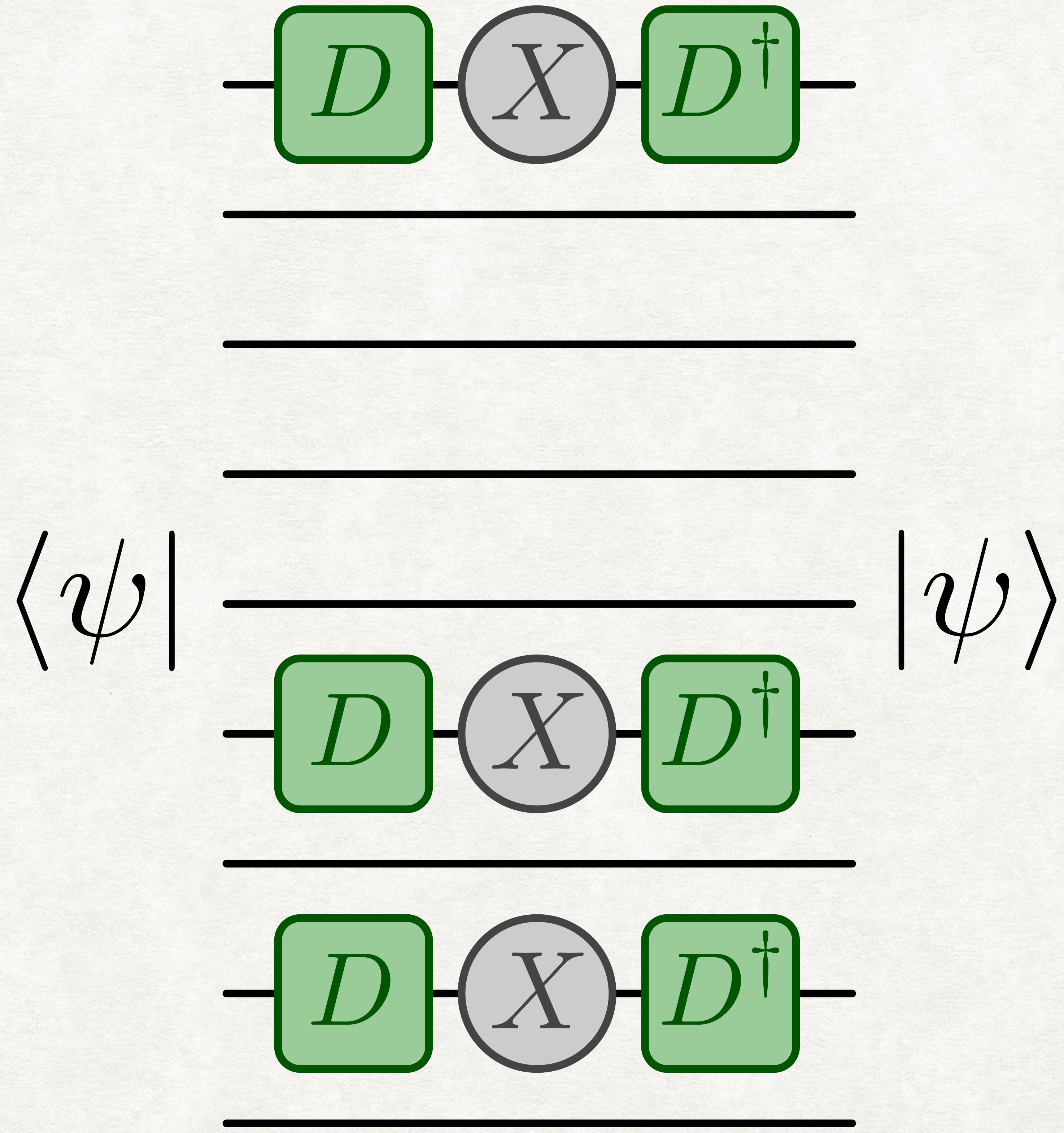


Local terms of $\tilde{H}$

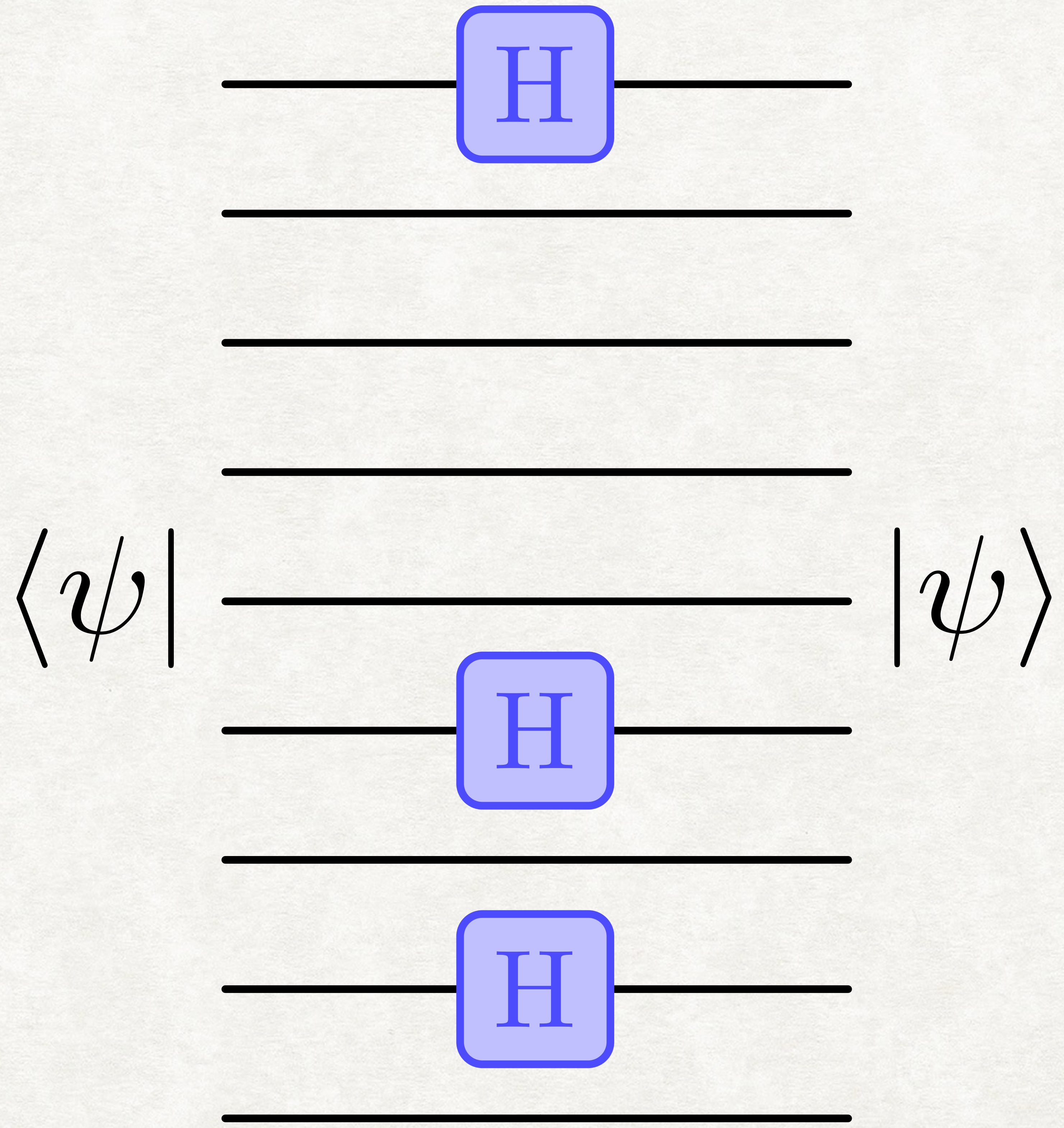
$$\tilde{H} \propto D^{\otimes n}\left(\sum_i \frac{I - X^{\otimes k}}{2}\bigg|_{A_i}\right)D^{\dagger\otimes n}$$



Local terms of $\tilde{H}$

$$\tilde{H} \propto \sum_i \frac{I - \mathrm{H}^{\otimes k}}{2}\Big|_{A_i}$$

Local energy: $E_i = \frac{1}{2}\left(1 - \langle\psi|\,\mathrm{H}^{\otimes k}{}_{A_i}\,|\psi\rangle\right)$



$\{\bullet\} = A$

Rotating CSS ⇒ NLACS

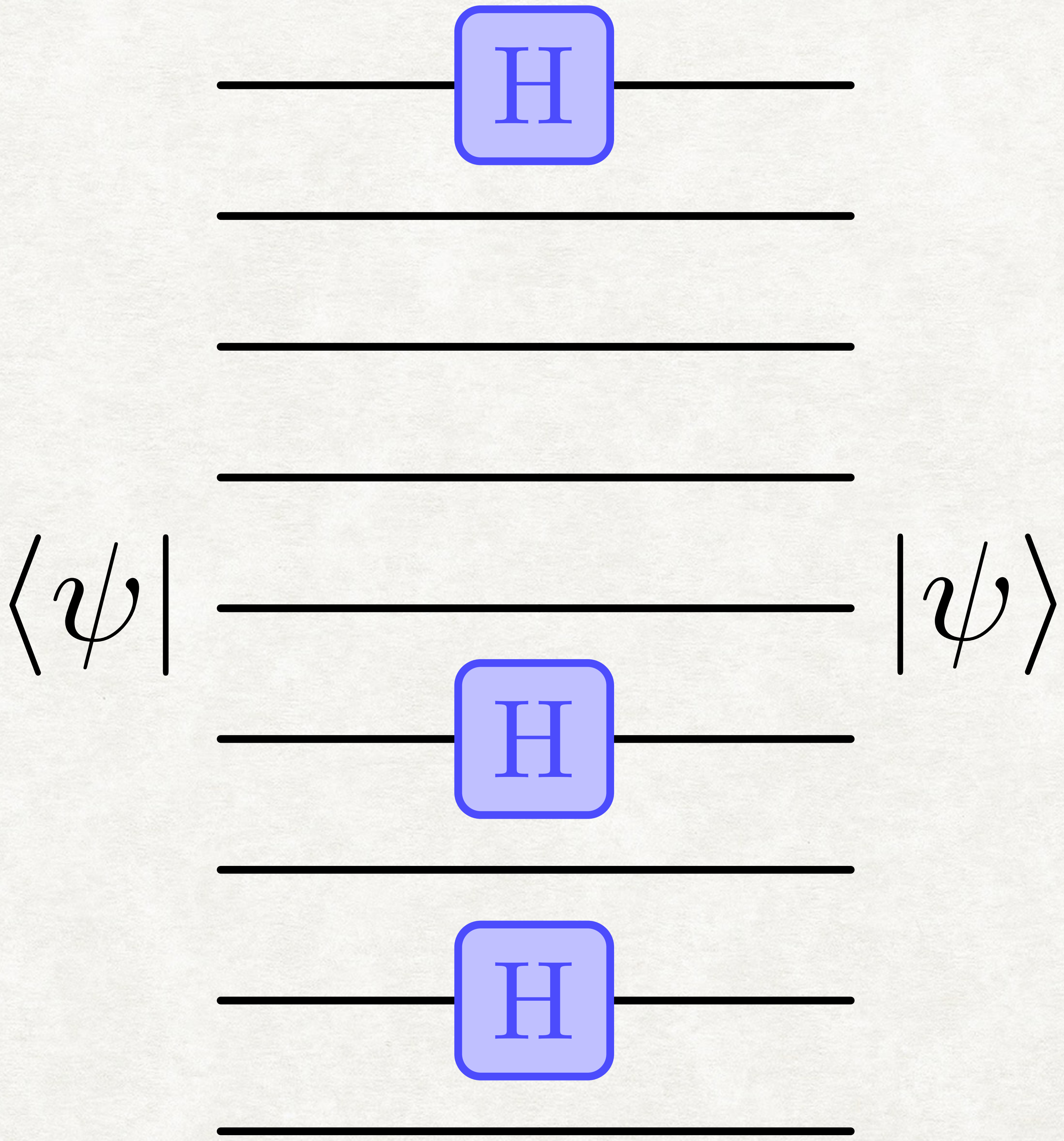
Proof components.

$|\psi\rangle$, arbitrary state

1. A local condition of $Stab(|\psi\rangle)$ at A implies an energy bound on the term.

2. $Stab(|\psi\rangle)$ satisfies this for $\Theta(m)$ terms of $\tilde{H} \equiv D^{\otimes n} H D^{\dagger \otimes n}$ if it is sufficiently large.

Combined ⇒ $\tilde{H}$ is NLACS



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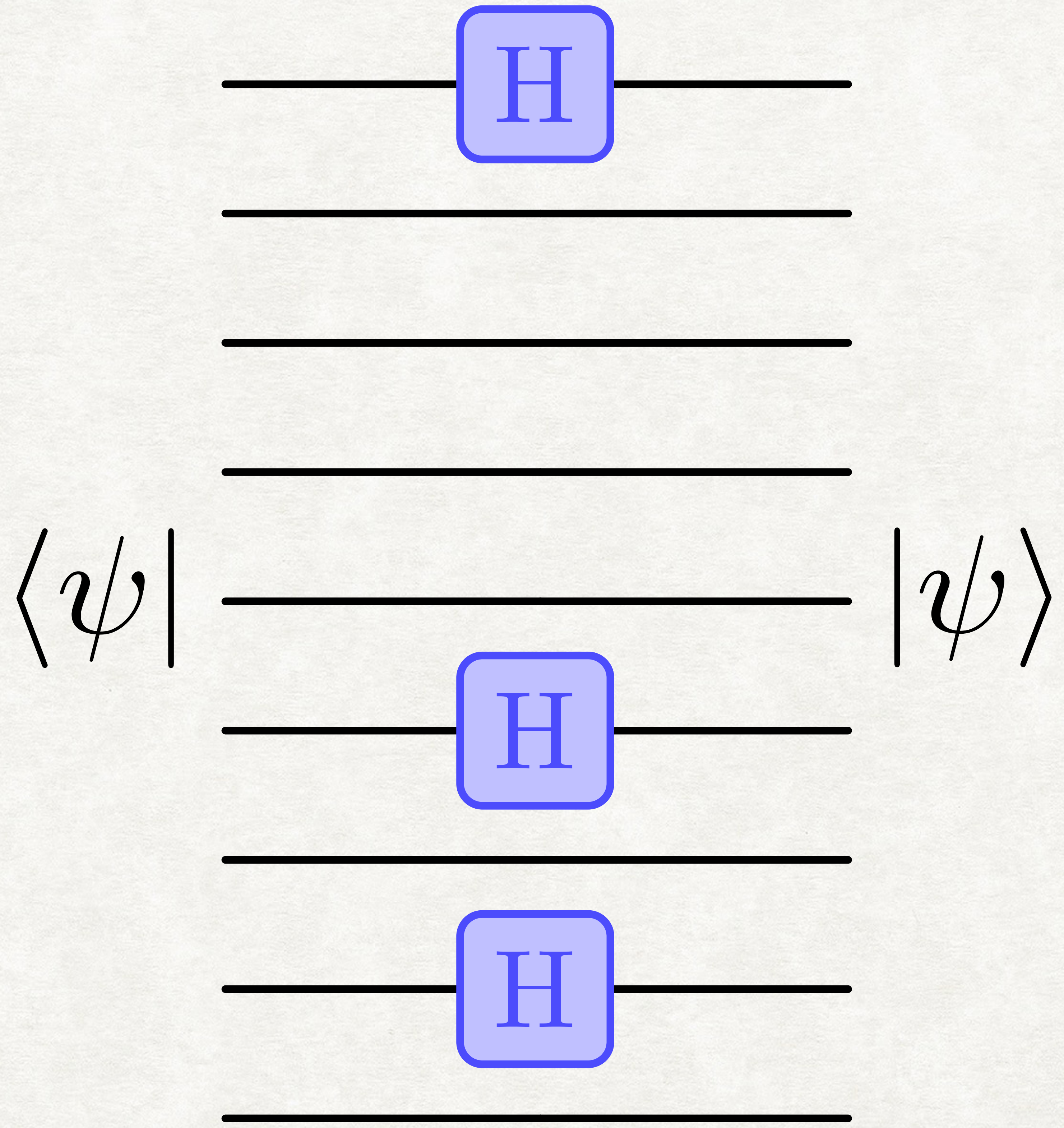
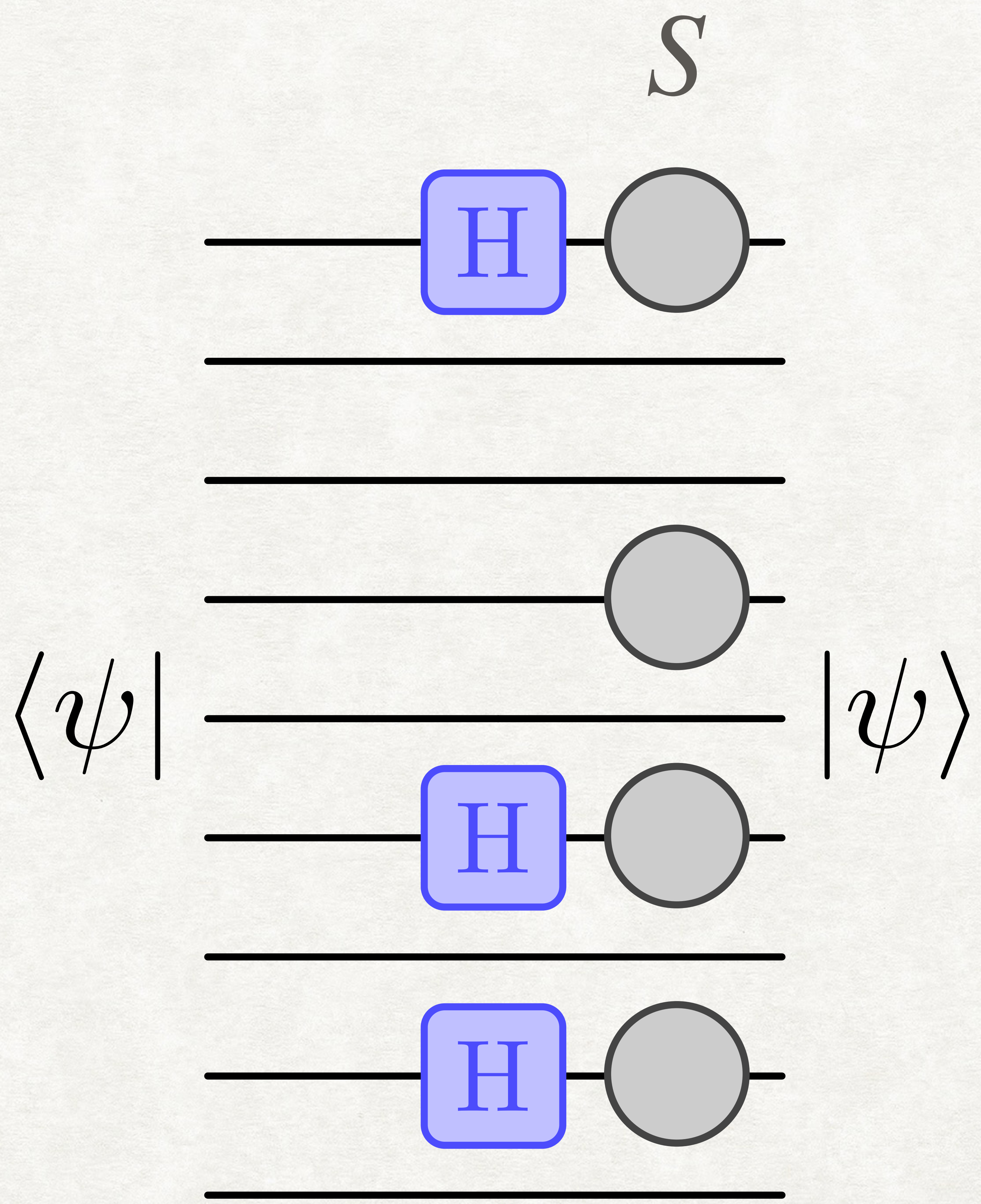


$\{\bullet\} = A$

Local bound — single term

Suppose $S \in Stab(|\psi\rangle)$ acts non-trivially at A

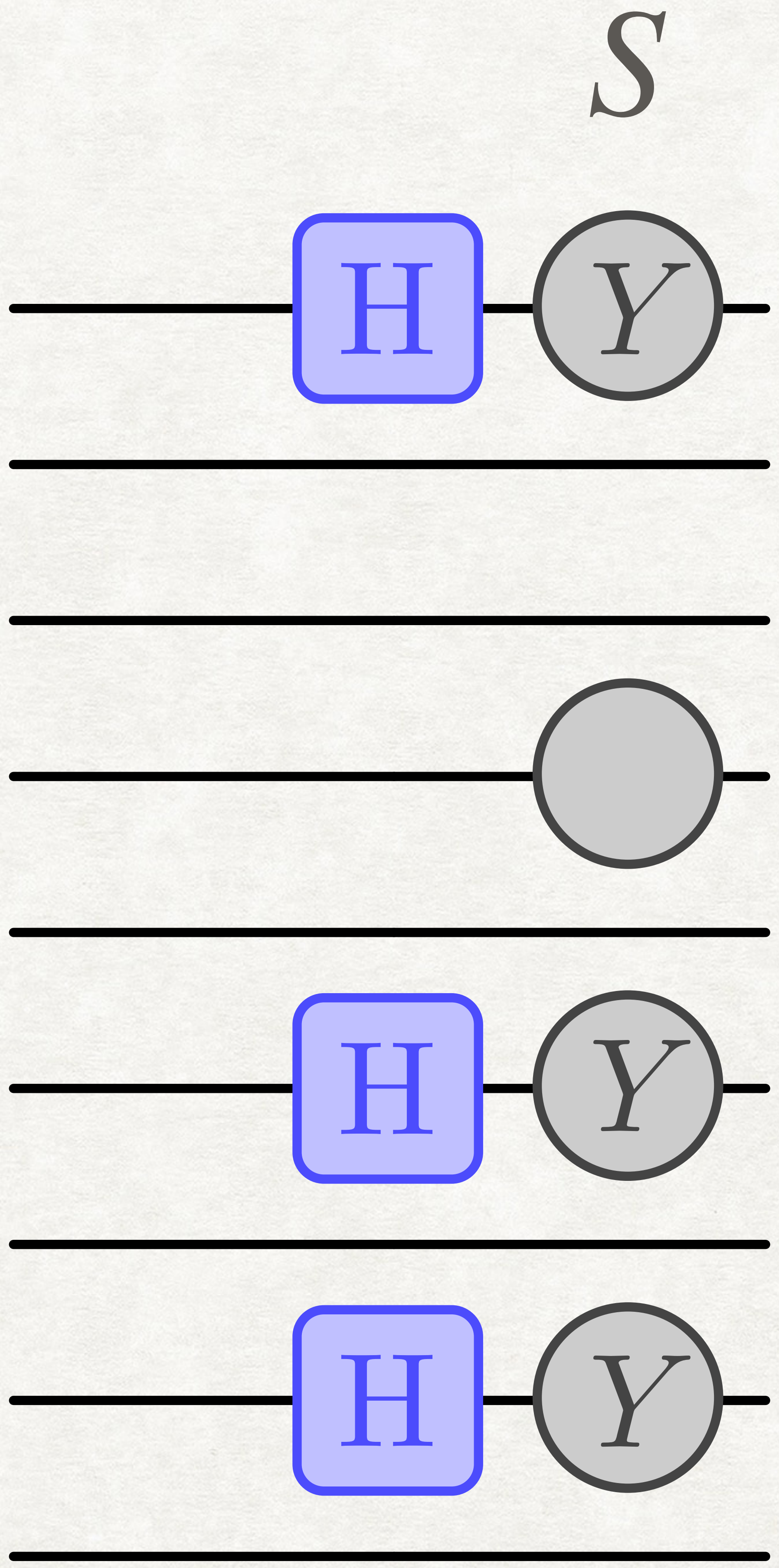


$\{\bullet\} = A$

Local bound — single term

Suppose $S \in Stab(|\psi\rangle)$ acts non-trivially at A

If the overlap has an odd # of Y's:



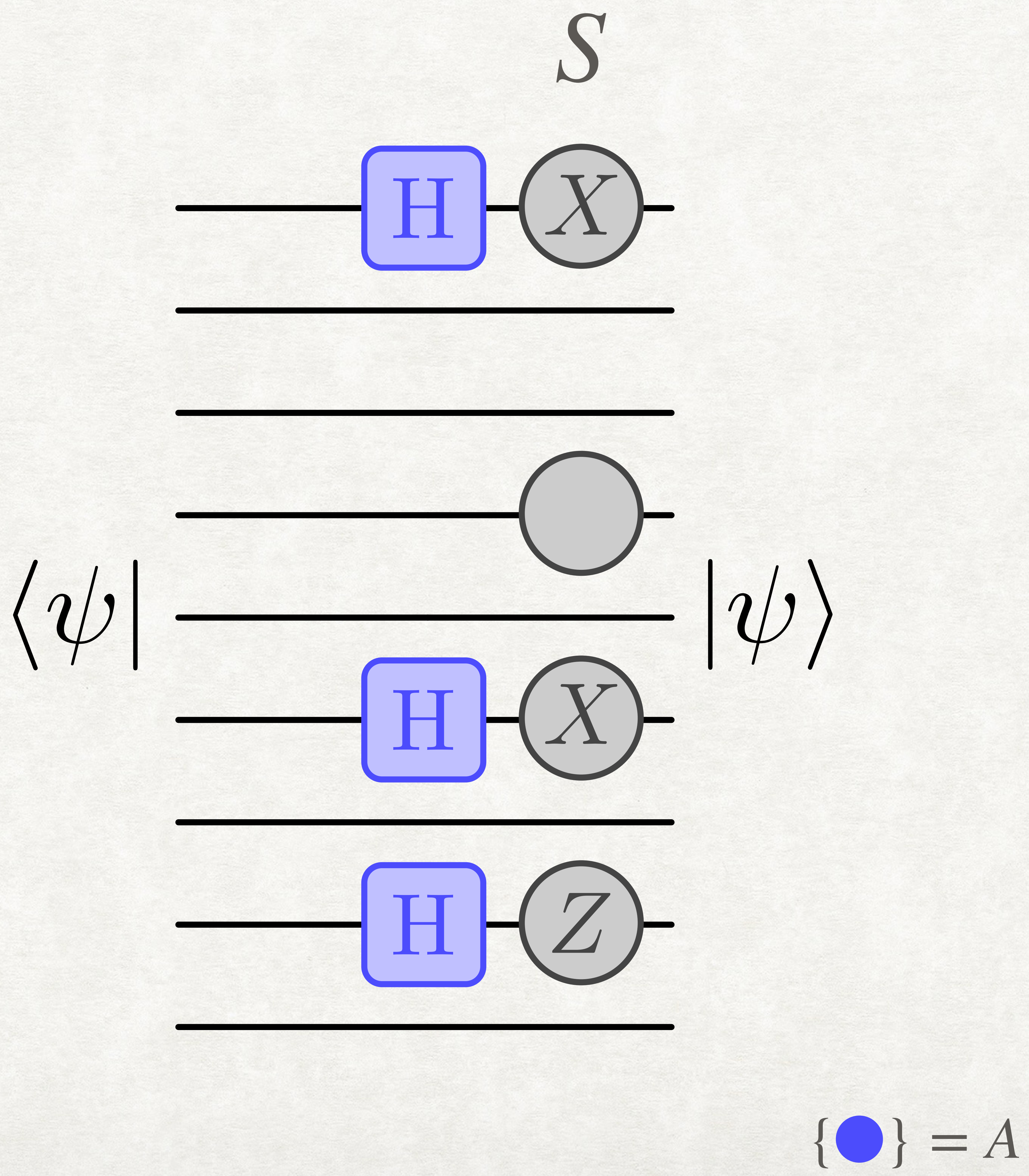
$$\langle\psi| \; S \; |\psi\rangle = 0$$

$\{\bullet\} = A$

Local bound — single term

Suppose $S \in Stab(|\psi\rangle)$ acts non-trivially at A

If the overlap has an odd total # of X's and Z's

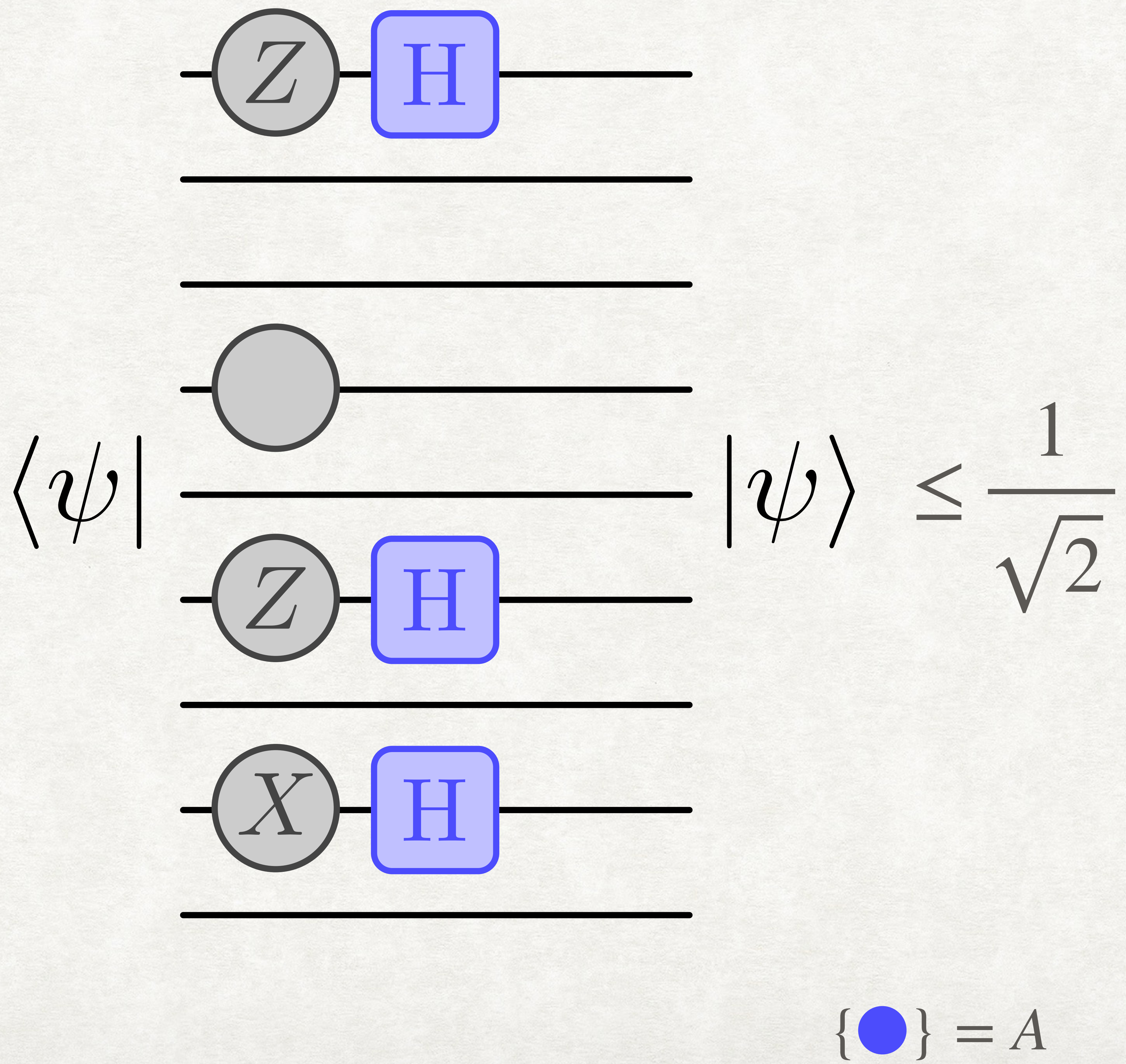


Local bound — single term

Suppose $S \in Stab(|\psi\rangle)$ acts non-trivially at A

If the overlap has an odd total # of X's and Z's then $|\psi\rangle$ and $(\mathrm{H}^{\otimes k})_{A_i}|\psi\rangle$ have anti-commuting stabilizers.

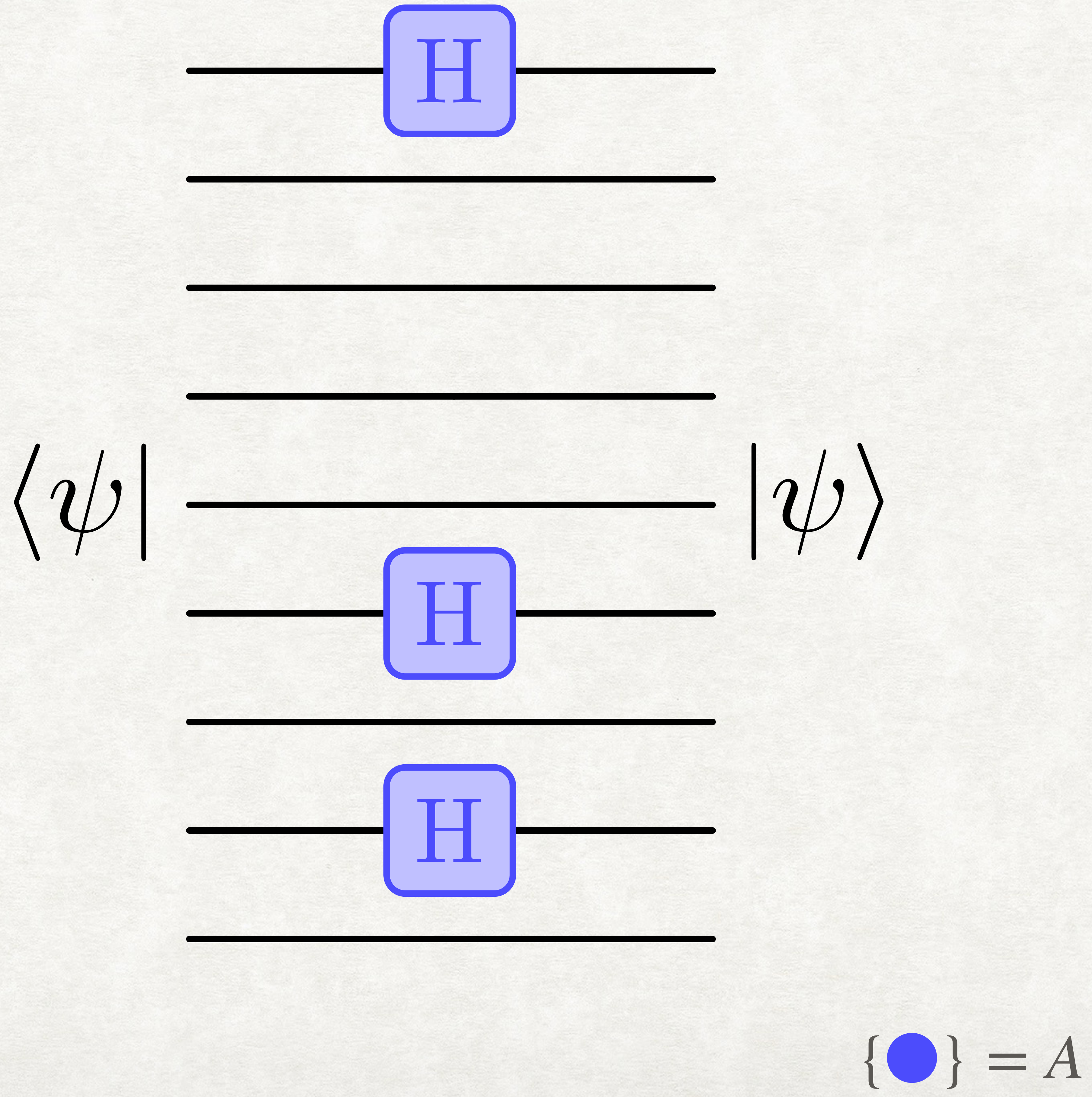
By Fact 2 the bound holds.



Local bound — single term

Suppose $S \in Stab(|\psi\rangle)$ acts non-trivially at A

How to guarantee these happen?



Local views and locally-commuting sets

$$P = P_1 \otimes P_2 \otimes P_3 \otimes \cdots \otimes P_n$$

Local view of P at qubits 1 and 2:

$$\rho_{\{1,2\}}(P) = P_1 \otimes P_2 \otimes \ I \ \otimes \cdots \otimes \ I$$

Local views and locally-commuting sets

Local view of P at $A \subseteq [n]$: $$\rho_A(P) = \begin{cases} P_i & \text{if } i \in A \\ I & \text{if } i \notin A \end{cases}$$

Def. $S \subseteq \mathscr{P}_n$ is **locally-commuting at** $A \subseteq [n]$ if $\rho_A(S)$ is a commuting group.

e.g.

$$\begin{matrix} \color{red}{I} & \color{red}{I} & X & I \\ \color{red}{I} & \color{red}{X} & Z & Y \\ \color{red}{X} & \color{red}{I} & Z & X \\ \color{red}{X} & \color{red}{X} & Y & I \end{matrix}$$

Pseudo-stabilizer property

Def. $|\psi\rangle$ is a **pseudo-stabilizer state at** $A \subseteq [n]$ if there is a subset $S \subseteq Stab(|\psi\rangle)$ s.t.

1. S is locally-commuting at A

2. $|\rho_A(S)| = 2^{|A|}$ (max possible size)

Local bound

Lemma. If $|\psi\rangle$ is pseudo-stabilizer at A, then $\rho_A(S)$ contains either:

1. A term with an odd # of Y's

2. A term with a total odd # of X's and Z's

Corollary. If $|\psi\rangle$ is pseudo-stabilizer at A, then $\langle\psi| \frac{I - \mathrm{H}^{\otimes k}}{2}\Big|_A |\psi\rangle \geq \sin^2\left(\frac{\pi}{8}\right)$.

Global bound — how many terms?

Proof components.

$|\psi\rangle$, arbitrary state

1. If $|\psi\rangle$ is pseudo-stabilizer at A, then there is a local energy lower bound.

2. $Stab(|\psi\rangle)$ satisfies this for $\Theta(m)$ terms of $\tilde{H} \equiv D^{\otimes n} H D^{\dagger \otimes n}$ if it is sufficiently large.

How to guarantee this for an almost-Clifford state?

Combined $\Rightarrow$ $\tilde{H}$ is NLACS

Global bound — how many terms?

Lemma. $|\psi\rangle$, prepared by $\leq cn$ T gates for $c \in (0,1) \Rightarrow |\psi\rangle$ is pseudo-stabilizer at $\Omega(n)$ local terms of $\tilde{H} \equiv D^{\otimes n} H D^{\dagger \otimes n}$.

Proof idea.

1. [Sort of] Trivial: upper bound on T-count gives *lower bound* on size of $Stab(|\psi\rangle)$.

2. *Upper bound* on size of $Stab(|\psi\rangle)$ in terms of sizes of locally-commuting subsets

Thus, large stabilizer group ⇒ many large locally-commuting subsets

Local + global

Lemma. If $|\psi\rangle$ is pseudo-stabilizer at $A \Rightarrow \langle\psi| \frac{I - \mathrm{H}^{\otimes k}}{2}\Big|_A |\psi\rangle \geq \sin^2\left(\frac{\pi}{8}\right)$.

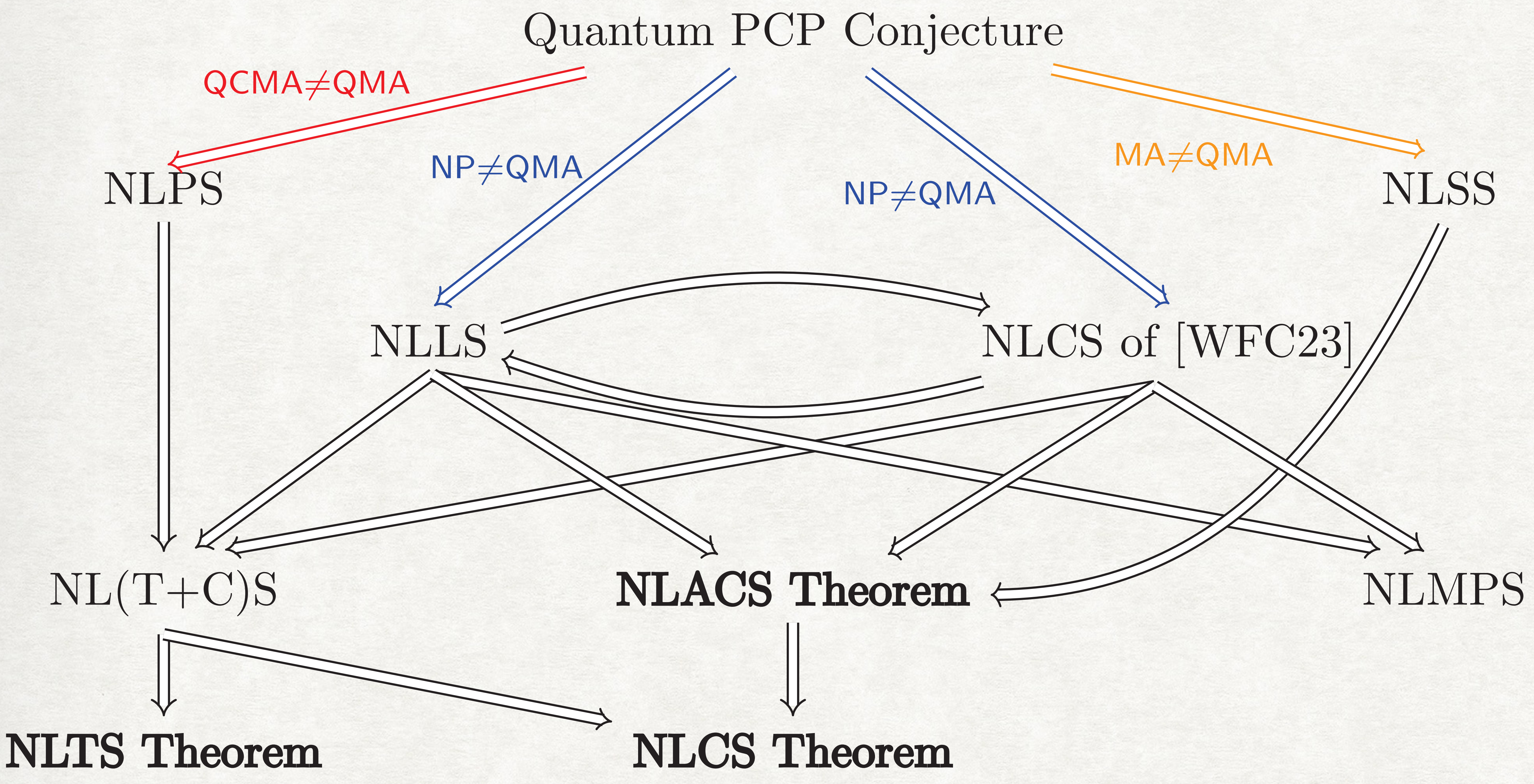
Lemma. $|\psi\rangle$, prepared by $\leq cn$ T gates for $c \in (0,1) \Rightarrow |\psi\rangle$ is pseudo-stabilizer at $\Omega(n)$ local terms of $\tilde{H} \equiv D^{\otimes n} H D^{\dagger \otimes n}$.

Theorem. $|\psi\rangle$, prepared by $\leq cn$ T gates for $c \in (0,1) \Rightarrow \langle\psi|\tilde{H}|\psi\rangle = \Omega(1)$

Outline

- Quantum complexity basics
- Implications of QPCP
- Simple NLACS Hamiltonian
- CSS Hamiltonians and joint NLTS/NLACS
- Future directions

"State lower bounds"



Ruling out classes of witnesses for LH-$\Omega(1)$

"Complexity lower bounds"

- New hardness results for LH-$\Omega(1)$, BQP-hardness, MA-hardness, etc.

More directions

- Primitives for QPCP:
  - Quantum locally-testable codes (QLTCs)
  - Quantum PCPs of Proximity (QPCPPs) or QPCPs of weaker soundness/locality
- Variants of QPCP:
  - QCPCP Conjecture [Weggemens, Folkertsma, Cade 23]
  - $QPCP_1$ Conjecture, i.e., perfect completeness (gap amplification for Clique Homology?)

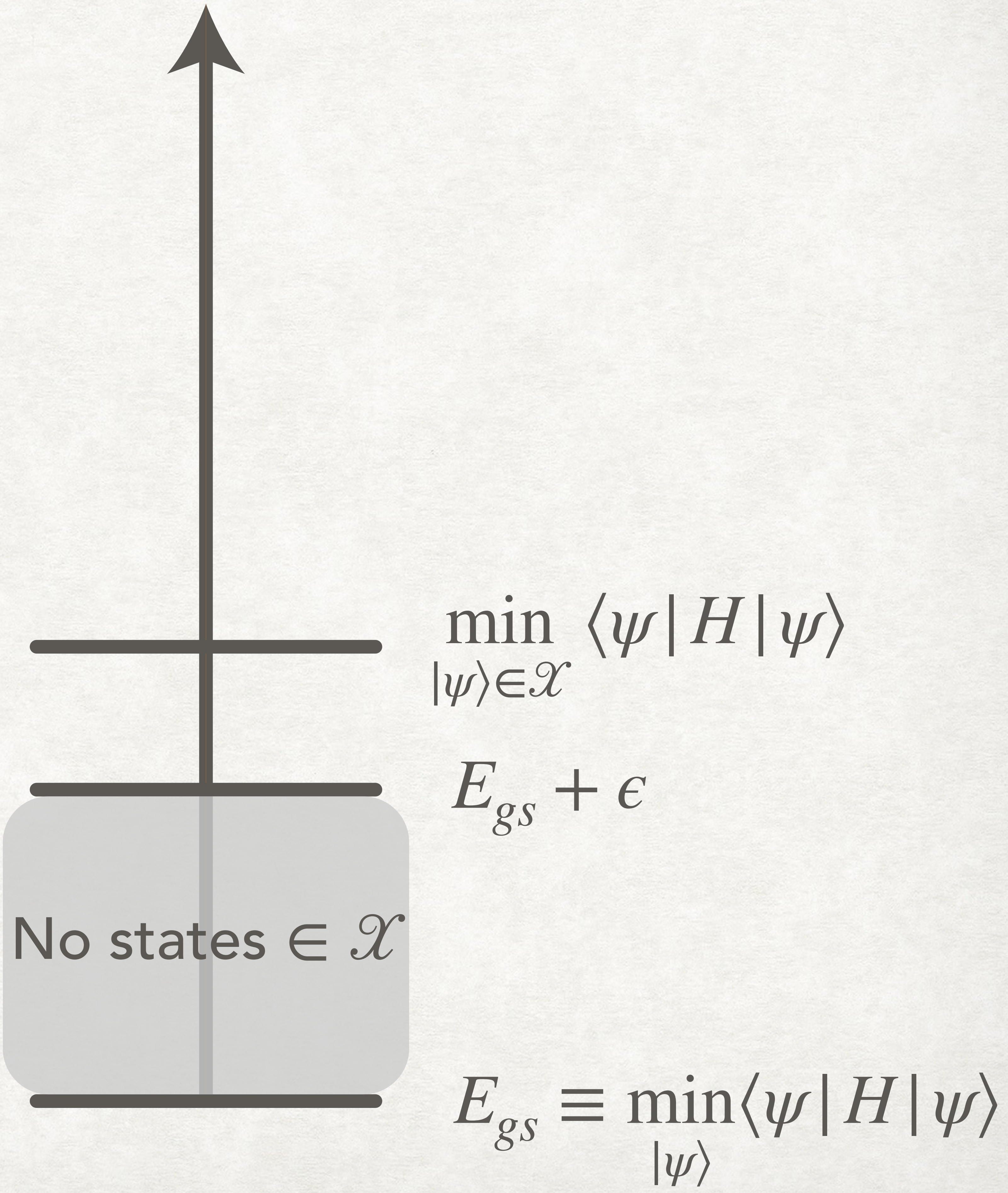
Recap

Energy estimation in NP/MA implies a special type of state

If QPCP is true and QMA≠NP,MA then these states can't be in the low-energy space of arbitrary local Hamiltonians.

NLTS, NLCS, NLACS verify that some of these are true

NLSS, NLLS are still open



Thanks!

Energy estimation in NP/MA implies a special type of state

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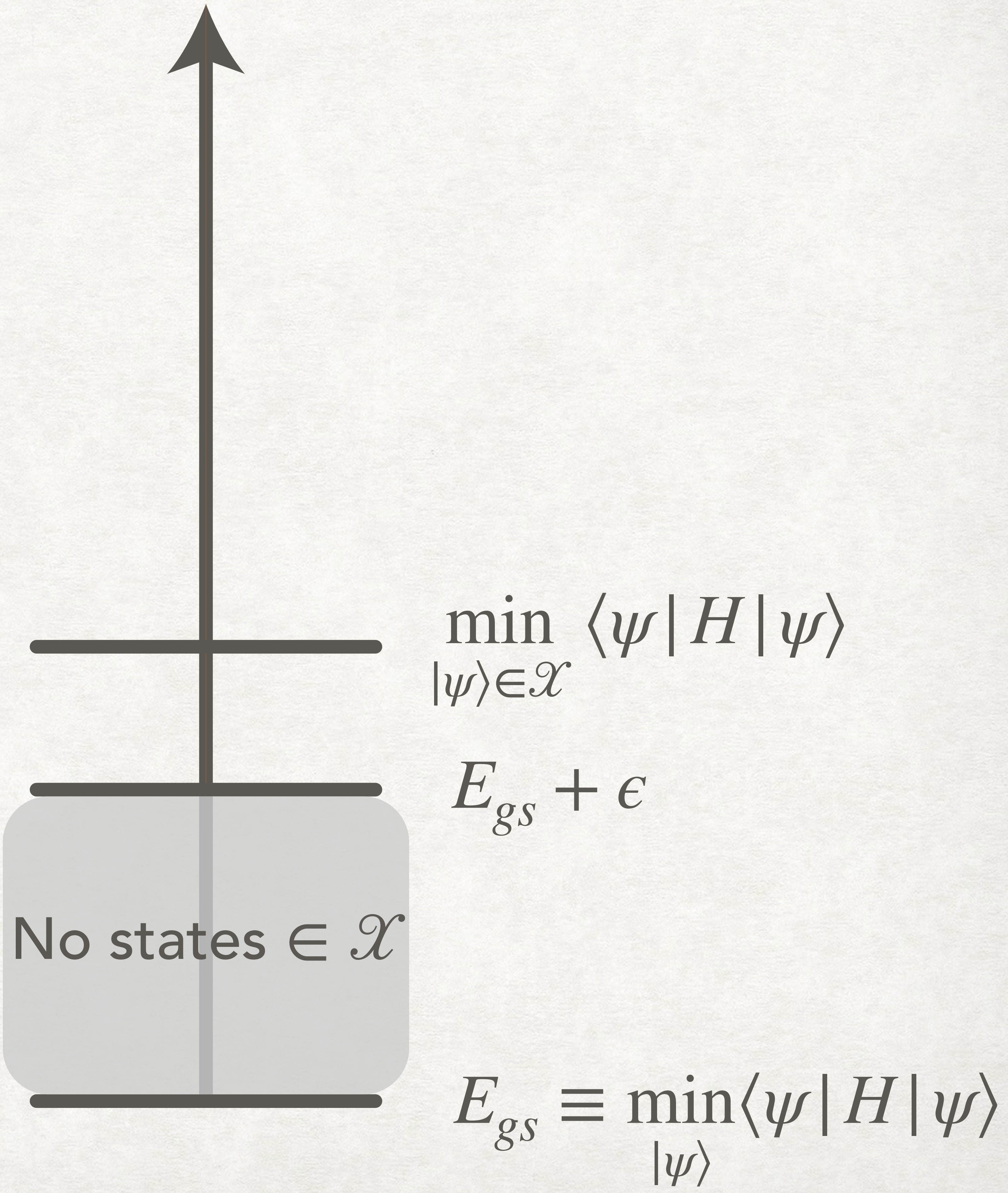

NLLS

NP energy estimation for trivial, stabilizer, almost-Clifford share a similar theme.

A state is k-locally approximable if:

(1) $|\psi\rangle$ has an *efficient* classical description, $desc(|\psi\rangle) \in \{0,1\}^{poly(n)}$.

(2) There is an efficient classical algorithm, W, which computes all k-reduced states of $|\psi\rangle$, i.e., for all $A \subseteq_k [n]$

$$\left| W\big(A, desc(|\psi\rangle)\big) - Tr_{-A}[|\psi\rangle\langle\psi|] \right| \leq \epsilon = O(1)$$

NLLS

NP energy estimation for trivial, stabilizer, almost-Clifford share a similar theme.

$|\psi\rangle$, k-locally approximable ⇒ energy estimation in NP

No low-energy locally-approximately states — NLLS Conjecture [CCNN23a, WFG23]

Question. Does an NLLS Hamiltonian exist?

NLSS

Candidate Hamiltonian: $CH_0C^\dagger \equiv \frac{1}{n}\sum C^\dagger|1\rangle\langle 1|_i C$ where C is a family of Haar-random, constant-depth circuits.

Ground state = $C|0\rangle^{\otimes n}$.

Theorem [HE23]. $C|0\rangle^{\otimes n}$ is not sampleable unless the Polynomial Hierarchy collapses.

Question. Are low-enough energy states of $CH_0C^\dagger$ also not sampleable?

NLSS

Rotated QLTC Hamiltonian:

Let H be a QLTC Hamiltonian and C, Haar-random constant-depth circuit.

Question. Are ground states of $CHC^\dagger$ not sampleable?

Question. If $|\varphi\rangle$ is not sampleable and $dist(|\psi\rangle, |\varphi\rangle) \leq \epsilon n$, is $|\psi\rangle$ not sampleable?

Combined $\Rightarrow$ rotated QLTC is an NLSS Hamiltonian

QLTCs

A QLDPC Hamiltonian, H, corresponds to a good *quantum locally-testable code* (QLTC) if $\langle\psi|H|\psi\rangle \geq dist\big(|\psi\rangle, gs(H)\big)/n$.

Question. Do good QLTCs exist?

Fact [EH17]**.** If H corresponds to a QLTC then H is an NLTS Hamiltonian.